Chapter 1

Mechanics. Oscillations.

1.1 Quantities, Principles and Fundamental Laws in Classical Mechanics

1.1.1. Notions and Basic Quantities in Classical Mechanics

Mechanics studies the simplest form of motion of the matter, the mechanical motion. The mechanical motion is the motion that causes the change in the position of the bodies, ones relative to the others or of their parts in space and in time.

Classical Newtonian mechanics studies the motion of the bodies that have much slower speeds than the speed of the light in vacuum, $c=3\cdot 10^8$ m/s.

Under classical mechanics, there is a series of fundamental laws (principles): the law of inertia or of the conservation of the impulse, the law of motion of a material point or of variation of the impulse, the law of reciprocal actions, the law of superposition of the forces, the law of gravitation. There are other laws besides the fundamental ones, like those concerning material: the law of elasticity, the law of friction etc.

The principle represents a statement suggested by an observation which meets the condition that all the consequences that result from its acceptance do not contradict the observation.

The pattern from which the classical mechanics develops is based on the causality principle and states that, under given initial conditions, a physical process always develops in a particular way. To the same causes there correspond the same effects.

The law represents a mathematical relationship which relates different physical quantities amongst themselves.

The main physical quantities presented in this chapter are: speed, acceleration, force, impulse, moment of force, angular momentum, kinetic energy, potential energy, mechanical work.

Kinematics is that part of mechanics that establishes the mathematical equations describing the motion of the bodies, disregarding the cause of the motion.

Dynamics is concerned with the study of the causes that produce the motion and establishes the mathematical equations that describe the motion.

In mechanics there is introduced the notion of material point.

By material point we understand a body of which dimensions can be neglected when its motion is studied. The motion is studied with respect to a referential system, arbitrarily chosen, as there is no absolutely fixed system of reference relating to which all motions can be studied. Hence, the motion and the rest are relative.

The reference system can be inertial or non-inertial. The inertial reference system is the system that has a rectilinear uniform motion or is at relative rest. The non-inertial reference system has an accelerated motion. Under classical mechanics and special relativity, the inertial reference systems are considered.

To the reference system there is rigidly attached a reference frame. The motion of a moving body is univocally determined if, each moment, its coordinates are known in relation with the reference system chosen. In mechanics there are especially used:

1) Cartesian coordinate system
2) spherical coordinate system

In the Cartesian coordinate system the position of a point *P* is given by the Cartesian coordinates *x, y* and *z* (Fig. 1.1). The vector $\vec{r}$, that connects the origin with point *P*, is called position vector or radius vector.

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad . \tag{1.1}$$

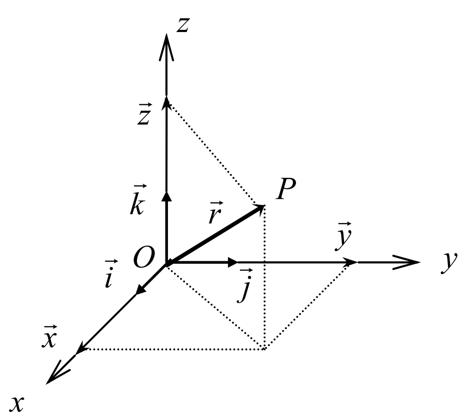


Fig. 1.1

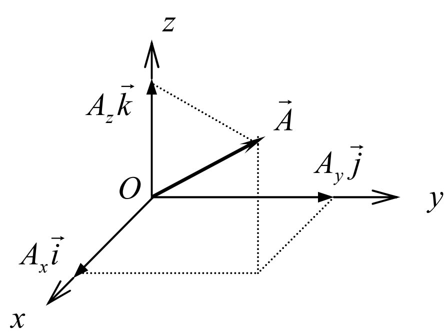


Fig. 1.2

In general, any arbitrary vector A can be written:

$$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k} \ , \tag{1.2}$$

where A_x, A_y and A_z are called the components of vector $\vec{A}$ (Fig. 1.2). According to the components, the length (*magnitude*) of vector $\vec{A}$ is:

$$A = \sqrt{\vec{A} \cdot \vec{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{1.3}$$

$\vec{i}$ $\vec{j}$ and $\vec{k}$ are unit vectors of the coordinate axes. One knows they satisfy the relations:

$$\left|\vec{i}\right| = \left|\vec{j}\right| = \left|\vec{k}\right| = 1;$$

$$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1;$$

$$\vec{i} \cdot \vec{j} = \vec{i} \cdot \vec{k} = \vec{j} \cdot \vec{k} = 0$$

In the spherical coordinates system the position of a point P is given by the spherical coordinates r, θ, φ (Fig. 1.3).

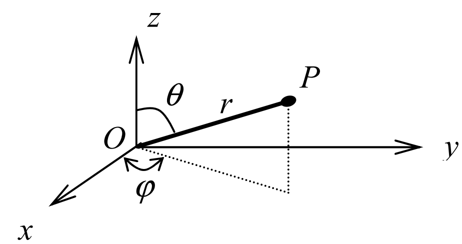


Fig. 1.3

Let $Oxyz$ be a Cartesian coordinates system against which the motion of a material point P is studied. The motion can be determined if the law of variation with respect to time of coordinates x, y, z of the point and the initial conditions are known:

$$x = x(t), \qquad y = y(t), \qquad z = z(t) \tag{1.4}$$

The relations (1.4) are equivalent to the relations indicating how a position vector varies with time:

$$\vec{r} = \vec{r}\,(t), \qquad \vec{r}\,(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}\,. \tag{1.5}$$

The relations (1.4) or (1.5) represent the *law of motion* of the material point.

a) Speed

Consider a material point that moves along a trajectory. At the moment t_1 the material point is in position P_1 and at the moment t_2 is in P_2. We denote Δs – the length of the curve P_1P_2. The *average speed* is defined by the ratio between the space crossed by the moving body and the time needed for crossing the space:

$$\mathrm{v}_m \equiv \frac{\Delta s}{\Delta t}. \qquad (1.6)$$

The average speed gives us a rather vague idea about the motion of the moving body. In order to increase precision, there needs to be introduced the notion of speed at a particular time, also called *instantaneous speed*. In point P_1, the position vector is $\vec{r}\,(t_1)$ whereas in point P_2, it is $\vec{r}\,(t_2)$. Vector $\Delta\vec{r}$ in Fig. 1.4 represents the difference between them/ the displacement. Δr is the curve P_1P_2. The ratio $\Delta\vec{r}\,/\,\Delta t$ is a collinear vector with curve P_1P_2 and defines *velocity*. Velocity is the rate of displacement. The magnitude of velocity is speed. If Δt tends towards 0, then P_2 tends towards P_1 and curve P_1P_2 tends towards the tangent in P_1. According to the definition of the derivative, we have:

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t}.$$

By definition, *instantaneous velocity* vector is:

$$\vec{\mathrm{v}} \equiv \frac{d\vec{r}}{dt} \qquad (1.7)$$

$d\vec{r}$ having the orientation of the tangent line at the curve. Hence, in each point of trajectory, velocity v has the orientation of the tangent line at the trajectory and its direction coincides with the direction of material point.

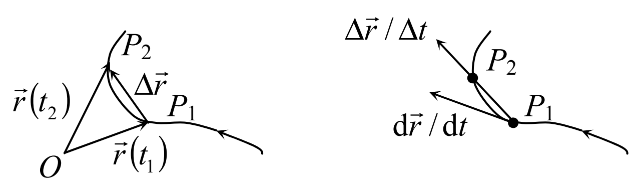


Fig. 1.4

Like any vector, velocity can be written as follows:

$$\vec{\mathrm{v}} = \mathrm{v}_x\vec{i} + \mathrm{v}_y\vec{j} + \mathrm{v}_z\vec{k}\ , \qquad (1.8)$$

where v_x, v_y, and v_z are the velocity components. From the relations (1.1) and (1.7) it results:

$$\vec{\mathrm{v}} = \frac{d\left(x\vec{i} + y\vec{j} + z\vec{k}\right)}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} \qquad (1.9)$$

By comparing the previous two relations, it results:

$$\mathrm{v}_x = \frac{dx}{dt},\ \mathrm{v}_y = \frac{dy}{dt}\ ,\ \mathrm{v}_z = \frac{dz}{dt} \qquad (1.10)$$

The length (*magnitude*) of velocity is:

$$\mathrm{v} = \sqrt{\mathrm{v}_x^2 + \mathrm{v}_y^2 + \mathrm{v}_z^2}. \qquad (1.11)$$

b) Acceleration

A change in velocity is called an acceleration. Acceleration is defined as the rate of change of velocity of an object with respect to time. Objects are only accelerated if a force is applied to them.

The instantaneous acceleration vector is defined by the relation:

$$\vec{a} \equiv \frac{d\vec{\mathrm{v}}}{dt} = \frac{d^2\vec{r}}{dt^2} \qquad (1.12)$$

In terms of acceleration components, we can write:

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}\ . \qquad (1.13)$$

From the relations (1.8) and (1.12) it results:

$$\vec{a} = \frac{d\mathrm{v}_x}{dt}\vec{i} + \frac{d\mathrm{v}_y}{dt}\vec{j} + \frac{d\mathrm{v}_z}{dt}\vec{k}\ . \tag{1.14}$$

By comparing the previous two relations, we get:

$$a_x = \frac{d\mathrm{v}_x}{dt} = \frac{d^2x}{dt^2}, a_y = \frac{d\mathrm{v}_y}{dt} = \frac{d^2y}{dt^2}, a_z = \frac{d\mathrm{v}_z}{dt} = \frac{d^2z}{dt^2} \tag{1.15}$$

We can compute the length (*magnitude*) of the acceleration vector using the following equation:

$$\vec{a} = \sqrt{a_x^2 + a_y^2 + a_z^2}. \tag{1.16}$$

Let $\vec{\tau}$ denote the tangent unit-vector. Considering $\vec{\mathrm{v}} = \mathrm{v}\,\vec{\tau}$, we can write the relation (1.12) as beside:

$$\vec{a} = \frac{d(\mathrm{v}\vec{\tau})}{dt} = \frac{d\mathrm{v}}{dt}\vec{\tau} + \mathrm{v}\frac{d\vec{\tau}}{dt}. \tag{1.17}$$

As $\vec{\tau}$ is a unitary vector, we have:

$$\vec{\tau} \cdot \vec{\tau} = 1. \tag{1.18}$$

By derivation with time, the relation becomes:

$$\frac{d\vec{\tau}}{dt}\vec{\tau} + \vec{\tau}\frac{d\vec{\tau}}{dt} = 0 \Rightarrow \vec{\tau}\frac{d\vec{\tau}}{dt} = 0. \tag{1.19}$$

This relation demonstrates the vectors $\vec{\tau}$ and $d\vec{\tau}/dt$ are perpendicular. Hence, $d\vec{\tau}/dt$ has the orientation of the normal line at the trajectory:

$$\frac{d\vec{\tau}}{dt} = \left|\frac{d\vec{\tau}}{dt}\right|\vec{n}, \tag{1.20}$$

$\vec{n}$ being the normal unit-vector. Consequently, relation (1.17) becomes:

$$\vec{a} = \frac{d\mathrm{v}}{dt}\vec{\tau} + \mathrm{v}\frac{d\tau}{dt}\vec{n}. \tag{1.21}$$

This relation demonstrates that the acceleration vector has two components, reciprocally perpendicular:

- a tangential component at the trajectory, a_t, determined by the speed variation with respect to time:

$$a_t = \frac{d\text{v}}{dt}, \tag{1.22}$$

- a normal component at the trajectory, a_n, determined by the speed variation with respect to direction

$$a_n = \text{v}\frac{d\tau}{dt}. \tag{1.23}$$

Considering that points P and P' (Fig. 1.5) are very close to each other, we can write :

$$d\alpha = \frac{d\tau}{\tau} = \frac{ds}{R} \Rightarrow \frac{d\tau}{dt} = \frac{1}{R}\cdot\frac{ds}{dt} = \frac{v}{R},$$

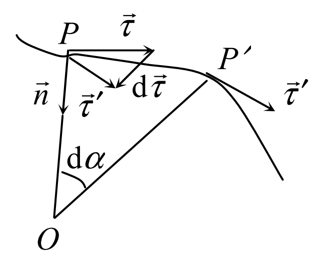


Fig. 1.5

where R is the curve radius of the trajectory in the proximity of point P. Hence, it results that :

$$\vec{a}_n = \text{v}\frac{d\tau}{dt}\vec{n} = \frac{\text{v}^2}{R}\vec{n} \tag{1.24}$$

By replacing (1.24) in (1.21), we obtain:

$$\vec{a} = \frac{d\text{v}}{dt}\vec{\tau} + \frac{\text{v}^2}{R}\vec{n}. \tag{1.25}$$

From this relation it results:

$$a_t = \frac{d\text{v}}{\text{dt}} \tag{1.26}$$

and

$$a_n = \frac{\mathrm{v}^2}{R}. \tag{1.27}$$

Particular situations

1. In linear motion, R→ ∞ and it results $a_n = 0$. If the linear motion is also uniform, v = *ct* and, accordingly, a_t is also null.
2. In circular uniform motion, $\mathrm{v} = ct \Rightarrow a_t = 0, a = a_n$.

c) Mechanical Work

The physical quantities can be classified as follows:

1) state quantities/parameters (state functions) depend only on the current state of the system, not on the way in which the system got to that state; for example: kinetic energy, potential energy, the internal energy of a thermodynamic system etc.;

2) process quantities/process functions which depend on the type of the process, on the path followed by the process/ are physical quantities that describe the transition of a system from an equilibrium state to another equilibrium state. As an example, mechanical work and heat are *process quantities* because they describe quantitatively the transition between equilibrium states of thermodynamic systems; for instance.

Let X denote any state quantity or process quantity. The infinitesimal variation of a state quantity is denoted by dX because it is an exact differential. For a process quantity, denotation δX is used. Since quantity X depends on the path followed, δX is not an exact differential. Consequently, let δL and δQ denote the elementary work and the quantity of elementary heat respectively.

Consider that upon a material point there is applied a force $\vec{F}$ that determines the displacement of a body along a trajectory. Generally,

- force can vary with respect to time, $F = F(t)$;
- force direction may not coincide with the direction of the material point displacement.

The elementary work done by force $\vec{F}$, that applies to the material point when it displaces its application point along the distance $d\vec{r}$, is defined by the following relation

$$\delta L = \vec{F} \cdot d\vec{r} \tag{1.28}$$

The total work done by force $\vec{F}$ when it displaces its application point from A to B, is:

$$L_{A\to B} = \int_A^B \vec{F} \cdot d\vec{r}. \tag{1.29}$$

d) Kinetic Energy

Kinetic energy is that part of the mechanical energy determined by the motion of a material point, generally of the body.

E_c or T denotes kinetic energy.

Force is equal to the impulse derivative with respect to time:

$$\vec{F} = \frac{d\vec{p}}{dt},\ \vec{p} = m\vec{v} \tag{1.30}$$

We calculate elementary work as follows:

$$\delta L = \frac{d\vec{p}}{dt} \cdot d\vec{r} = d\vec{p} \cdot \frac{d\vec{r}}{dt} = d(m\,\vec{v}) \cdot \vec{v} = d\left(\frac{m\,v^2}{2}\right). \tag{1.31}$$

Kinetic energy is defined by the relation:

$$T \equiv E_c \equiv \frac{m\mathrm{v}^2}{2}. \tag{1.32}$$

Kinetic energy (*energy of motion*) is that part of the mechanical energy determined by the motion of a material point, generally of the body.

From the previous two relations we obtain:

$$\delta L = dT. \tag{1.33}$$

In the case of the motion between *A* and *B*, by integrating relation (1.33), we get:

$$L_{A\to B} = \int_A^B dT = T(B) - T(A) = \frac{m\mathrm{v}_B^2}{2} - \frac{m\mathrm{v}_A^2}{2},$$

$$L_{A\to B} = \Delta T \tag{1.34}$$

Relation (1.34) is the mathematical expression of the theorem of the kinetic energy variation. *The theorem of the kinetic energy variation shows that the work done between two positions A and B is equal to the kinetic energy variation of the material point between the two positions.*

e) Conservative Forces

The conservative forces are the forces for which the work done between two points A and B (Fig. 1.6) does not depend on the path taken by the application point of the force, they are path-independent. In a more general sense, a conservative force is any force which may be expressed as a gradient of a scalar potential.

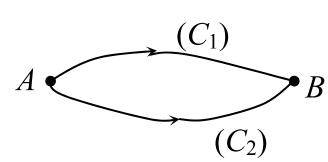


Fig. 1.6

$$\int_{A(C_1)}^{B} \vec{F}\cdot d\vec{r} = \int_{A(C_2)}^{B} \vec{F}\cdot d\vec{r}. \tag{1.35}$$

If the working point is displaced along the closed graph $A \rightarrow B \rightarrow A$, the work done vanishes.

$$L_{A\rightarrow B\rightarrow A} = 0. \tag{1.36}$$

Hence, the work done by a conservative force along a closed graph is zero:

$$\oint_{(C)} \vec{F} \cdot d\vec{r} = 0. \tag{1.37}$$

By using the Stokes-Ampère theorem, $\oint_{(C)} \vec{A} \cdot d\vec{r} = \iint_S rot\vec{A} \cdot d\vec{S}$, we obtain

$$\text{curl}\vec{F} = 0. \tag{1.38}$$

The curl of a conservative force vanishes. Examples of conservative forces are: gravitational force, electric force, elastic force.

The non-conservative forces are the forces for which the work done between two points depends on the path. For example: friction and magnetism.

f) Potential Energy

Potential energy is that part of mechanical energy that depends on the system configuration, which is on the position of the particles in a force field. Potential energy depends on the system coordinates. We can speak only of the variation of the potential energy. For instance, this variation can be turned into kinetic energy or vice versa.

As a result, the choice of the point where $U = 0$, called *the zero-point of the potential energy* or *reference point,* is arbitrary. In some cases, the choice of a certain zero-point of the potential energy is more convenient. The choice of any other zero-point of the potential energy gives the same result.

The potential energy can be uniquely defined only for the conservative forces. Let us consider material point in a conservative force field. If the curl of a vector field is null, then this vector field results from a scalar field by applying the gradient ($\mathrm{rot}\ \vec{A} = 0 \ \Rightarrow \ \vec{A} = \mathrm{grad}\ \varphi$) since $\mathrm{rot\ grad}\ \varphi = 0$, for any φ.

With the conservative forces we obtain $\mathrm{rot}\vec{F} = 0$.

By definition, the scalar function from which F results is the potential energy:

$$\vec{F} = -gradU \ , \tag{1.39}$$

and we have $\vec{F} = -\mathrm{grad}\,U = -\left(\frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k}\right)$.

We compute elementary work:

$$\delta L = \vec{F} \cdot d\vec{r} = -\left(\frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k}\right) \cdot \left(dx\vec{i} + dy\vec{j} + dz\vec{k}\right) =$$

$$= \left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz\right) = -dU$$

and we obtain

$$\delta L = -dU \tag{1.40}$$

If the application point of the force is disclosed from A to B, integrating the relation (1.40) we get:

$$L_{A\to B} = -\int_A^B dU = U(A) - U(B) \ . \tag{1.41}$$

Let point A be arbitrarily designate the zero-point of the potential energy, that is we have $U(A) = 0$, $L_{\mathrm{point\,ref.}\to B} = -U(B)$ and we get

$$U(B) = L_{B \to \text{point ref}} = \int_{\text{B}}^{\text{point ref.}} \vec{F} \cdot d\vec{r} \tag{1.42}$$

The potential energy of a material point, in any position B from a conservative force field, is equal to the work done by the forces of the field in order to disclose the material point from the considered position to the reference point.

Examples

1) *Elastic Potential Energy*

We take the equilibrium position as reference point: $U(0) = 0$.

$$U(x) = \int_x^0 F\,dx = -k\int_x^0 x\,dx = \frac{kx^2}{2},$$

and

$$U(x) = \frac{kx^2}{2} \tag{1.43}$$

2) *Electrical Potential Energy*

According to Coulomb's law, the interaction force between two electric charges Q and q is:

$$\vec{F} = \frac{Qq}{4\pi\varepsilon r^3}\vec{r} \tag{1.44a}$$

We take the reference point to infinity : $U(r = \infty) = 0$.

$$U(r) = \int_r^\infty \vec{F} \cdot d\vec{r} = \frac{Qq}{4\pi\varepsilon}\int_r^\infty \frac{1}{r^3}\vec{r} \cdot d\vec{r} = \frac{Qq}{4\pi\varepsilon}\int_r^\infty \frac{1}{r^2}dr = \frac{Qq}{4\pi\varepsilon r}. \tag{1.44b}$$

Considering that the electric potential at distance r from charge Q is given by the relation

$$V(r)=\frac{Qq}{4\pi\varepsilon r}, \tag{1.44c}$$

one gets:

$$U(r)=\frac{Qq}{4\pi\varepsilon r}=qV(r). \tag{1.44d}$$

3) *Gravitational Potential Energy*

According to the universal attraction law, the interaction force between two masses m_1 and m_2 (Fig. 1.7) is:

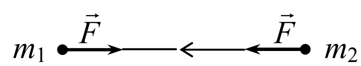


Fig. 1.7

$$\vec{F}=K\frac{m_1 m_2}{r^3}\vec{r}, \tag{1.45a}$$

where K is the universal gravitational constant. Let ∞ be the reference point: $U(r=\infty)=0$.

$$U(r)=\int_r^\infty \vec{F}\cdot d\vec{r}=K m_1 m_2\int_r^\infty \frac{1}{r^2}dr=-K m_1 m_2\frac{1}{r}, \tag{1.45b}$$

$$U(r)=-\frac{K m_1 m_2}{r} \tag{1.45c}$$

For the bodies in the gravitational field from Earth, it is convenient to have zero-point of the potential energy at the Earth surface (Fig. 1.8): $U(r = R) = 0$.

$$U(h)=\int_{R+h}^{R}\vec{F}\cdot d\vec{r}=-K\cdot m\cdot M\int_{R+h}^{R}\frac{1}{r^2}dr=K\cdot m\cdot M\left(\frac{1}{R}-\frac{1}{R+h}\right) \quad (1.45d)$$

$$U(h)=K\cdot m\cdot M\cdot\frac{h}{R(R+h)}\ . \quad (1.45e)$$

For low heights we have: $h << R$, $R(R+h) \cong R^2$,

$$U(h)=K\cdot m\cdot M\cdot\frac{h}{R^2}\ . \quad (1.45f)$$

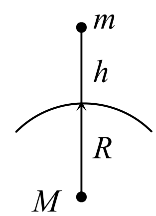


Fig. 1.8

At the Earth surface, the gravitational force is:

$$F=K\cdot\frac{mM}{R^2}=m\cdot g_0;\ \ g_0=K\frac{M}{R^2}, \quad (1.45g)$$

where g_0 is the gravitation acceleration at the Earth surface.

Potential energy becomes:

$$U(h)=m\cdot g_0\cdot h \quad (1.46)$$

1.1.2 Fundamental Principles of Classical Dynamics

1) *The Law of Inertia*

A material point tends to stay in a rectilinear uniform motion or at relative rest unless acted upon by an (external) force.

2) *The Fundamental Law of Dynamics*

In classical mechanics, mass is constant: $m = \mathrm{const}$.

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{\mathrm{v}})}{dt} = m\frac{d\vec{\mathrm{v}}}{dt} = m\vec{a}\,,$$

$$\vec{F} = m\vec{a}\,. \tag{1.47a}$$

Statement. If a force $\vec{F}$ acts upon a body, it applies an acceleration directly proportional to and having the same orientation with $\vec{F}$ and inversely proportional with its mass.

If $\vec{F} = 0 \Rightarrow \vec{a} = 0 \Rightarrow \vec{\mathrm{v}} = \mathrm{const}$.

By using the components of force $\vec{F}$, we obtain:

$$\vec{F} = m\frac{d^2\vec{r}}{dt^2}; \quad \vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}$$

$$F_x\vec{i} + F_y\vec{j} + F_z\vec{k} = m(\frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}). \quad (1.47b)$$

Identifying the coefficients of $\vec{\mathrm{i}}$, $\vec{j}$, and $\vec{\mathrm{k}}$ it results

$$F_x = m\frac{d^2x}{dt^2}, \quad (1.48)$$

$$F_y = m\frac{d^2y}{dt^2}, \quad (1.49)$$

$$F_z = m\frac{d^2z}{dt^2}. \quad (1.50)$$

The relations (1.48), (1.49) and (1.50) are *equations of motion*. They demonstrate that any motion can be decomposed in three linear motions by three perpendicular directions. The solutions to these equations represent *the law of motion* of the material point:

$$x = x(t), \qquad y = y(t), \qquad z = z(t). \quad (1.51)$$

By removing time from relations (1.51) we obtain the *equation of the trajectory*.

3) *The Law of Reciprocal Actions*

If one body acts upon another body with a force, the second one reacts with an equal force in magnitude but opposite in direction, called *reaction*.

4) *The Law of the Superposition of Forces*

If more forces act upon a material point, each force acts independently of the others.

1.1.3. Laws of Conservation

1) The Law of Conservation of Momentum

The mathematical expression is

$$\vec{F} = \frac{d\vec{p}}{dt}; \vec{F} = 0 \Rightarrow \vec{p} = \text{const.} \tag{1.52}$$

Statement. An applied force is equal to the rate of change of momentum.

2) The Law of Conservation of Angular Momentum

The angular momentum of a particle with respect to a fixed point, called origin, is defined by the relation:

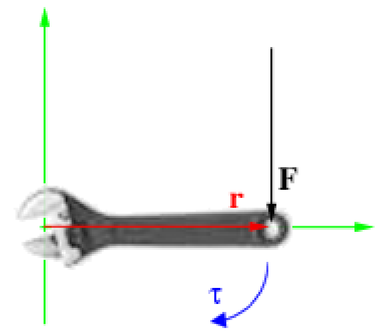
$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{\mathrm{v}}. \tag{1.53a}$$

For a circular motion, $\vec{r} \perp \vec{p}$ and we have:

$$L = m \cdot \mathrm{v} \cdot r \cdot \sin 90^\circ = m \cdot \mathrm{v} \cdot r. \tag{1.53b}$$

The moment of force (*torque*) $\vec{M}, \vec{\tau}$ with respect to the same fixed point is defined by the relation:

$$\vec{M} = \vec{r} \times \vec{F}. \tag{1.54}$$



Moment of force (torque)*

From the above, we derive relation (1.53*a*) with respect to time:

$$\frac{d\vec{L}}{dt}=\frac{d}{dt}\left(\vec{r}\times m\vec{v}\right)=\frac{d\vec{r}}{dt}\times m\vec{v}+\vec{r}\times m\frac{d\vec{v}}{dt}=\vec{v}\times m\vec{v}+\vec{r}\times m\vec{a}=\vec{r}\times\vec{F}=\vec{M},$$

and

$$\vec{M}=\frac{d\vec{L}}{dt}. \tag{1.55}$$

This relation is the mathematical expression of the *theorem of the angular momentum variation*. The statement of the theorem: the moment of force (*torque*) is equal to the variation with respect to time of the angular momentum or torque is the time-derivative of angular momentum.

From relation (1.55) it results:

$$\vec{M}=0\Rightarrow\vec{L}=\text{const.} \tag{1.56a}$$

The statement of the conservation law of the angular momentum: if the resulting moment of force (torque) vanishes, then the angular momentum is constant with respect to time. This happens, for instance, with the motion in a central force field. In such a field, in any point, the force is orientated along the position vector $\vec{r}$, if the origin of the coordinate axes is chosen in the center of the field. In this situation we have:

$$\vec{F} = f(r)\vec{r}, \quad \vec{M} = \vec{r} \times f(r)\vec{r} = 0. \tag{1.56b}$$

Examples of central force fields: gravitational field, electric field created by an electric charge, etc.

3) The Law of Conservation of Mechanical Energy

From the relations (1.33) and (1.40) one obtains:

$$\delta L = dT; \quad \delta L = -dU; \; dT + dU = 0; \quad d(T + U) = 0;$$

$$\boxed{T + U = \text{const.}} \tag{1.57}$$

The statement of the law of mechanical energy conservation: on condition conservative forces act upon a material point, the sum of the kinetic energy and potential energy is constant with respect to time.

The systems upon which only conservative forces act are called *conservative systems*.

The systems upon which non-conservative forces can act as well, are called *dissipative systems*. In this situation, the mechanical energy decreases in time, turning into other forms of energy, like thermal energy.

Views upon the Conservation Laws

The conservation laws are a result of the fact that Euclidian space is homogeneous and isotropic and time is uniform.

The free space is homogeneous, in other words, it does not differ from one point to another. If a figure is displaced, without rotation, from one place to another, no change in its size or its geometrical properties occurs. Furthermore, the physical properties (inertia, internal forces) of a body do not change if that body is displaced to another point in space. Therefore, the geometrical and physical properties are invariant with the space displacement of the object.

The space is isotropic, that means that all the directions are equivalent. The geometrical and physical properties of an object do not change if we rotate it in space. That is to say the geometrical and physical properties are invariable with rotation.

Time is uniform. In other words, the laws of motion of any given system apply independently from the origin of time. For instance, Coulomb's law or the gravitational law is the same at any time.

The space homogeneity determines the conservation of momentum. The space isotropy leads to the conservation of the angular momentum as the time uniformity leads to the energy conservation.

1.2. Oscillations

1.2.1. Classification

A system is in stable equilibrium on condition it has minimum potential energy. If the system is displaced from the stable equilibrium position, there applies a force that tends to bring back the system into its initial position. For short displacement from the equilibrium position, this force is elastic: $F = -kx$.

The oscillatory motion is the motion of a body, of a system, from one side to the other of the equilibrium position. A more general definition is the following:

The oscillatory motions are those motions that repeat periodically or quasi-periodically in time.

A *harmonic oscillator* consists in a system which, when displaced from its equilibrium position, experiences a restoring force F proportional to the displacement x, $F = -kx$ where k is a positive constant, the spring constant.

When there is a single force F acting on the system, the system is called a *simple harmonic oscillator*. The system undergoes *simple harmonic motion*: *sinusoidal oscillations* about the equilibrium point, A constant *amplitude* is characterizing the motion and there is also a constant *frequency* (which does not depend on the amplitude). *Amplitude* is maximal displacement from the equilibrium. *Frequency* represents the number of cycles the system performs per unit time, and is $\nu = \frac{1}{T}$.

In the presence of a frictional force (damping) proportional to the velocity, the harmonic oscillator is described as *damped*. In this case, the frequency of the oscillations is smaller than in the non-damped case. The amplitude of the oscillations is not constant, and decreases with time.

If an external time-dependent force is acting on the oscillator, the harmonic oscillator is described as *driven*.

Examples of oscillations: pendula (with small angles of displacement), masses connected to springs and acoustical systems, electrical harmonic oscillators (an LC circuit, a RLC circuit).

In periodic oscillation, the values of all physical quantities characteristic of the oscillation process repeat at equal time intervals. The minimum time interval after which this value repeats is called *period*, *T*. Period represents the duration of a complete oscillation or period is the number of cycles as a result of time (time/cycle) or the time it takes the system to complete an oscillation cycle. Period is also the inverse of frequency. The (*oscillation*) *frequency*, v, is defined by the relation $v = 1/T$ and represents the number of the complete oscillations done within the time unit.

In quasi-periodic oscillations, only a part of the physical quantities get values that repeat at equal time intervals.

According to the physical character of the oscillation, there are:

a) mechanical oscillations, when the kinetic energy is turned into potential energy and vice versa, like the oscillations of a pendulum, vibrations of a string, of a membrane etc.;

b) electromagnetic oscillations, when electric energy turns into magnetic energy and vice versa, like the oscillations from an oscillation circuit

c) electromechanical oscillations, when electric energy turns into mechanical energy or vice versa.

1.2.2. Non-Damped/Simple Harmonic Oscillations

This type of motion is generated in an elastic force field, in the absence of friction. The object (system) is displaced from its equilibrium position and left free. Let us consider a mass *m*, undergoing such type of motion along axis *Ox*, and the origin of the coordinate system which identifies with the equilibrium position. Displacement of mass *m* about the equilibrium position at a given point is x. The simple harmonic oscillator has no driving force, and no friction (damping), so the oscillator experiences the elastic force:

$$F = -k\,x, \tag{1.58}$$

where *k* is the spring constant, hence the motion equation of the material point. According to the fundamental law of dynamics, one has:

$$\vec{F} = m\vec{a} = m\left(\frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}\right) \tag{1.59}$$

As $y = 0$, $z = 0$, one gets:

$$F = m\frac{d^2x}{dt^2}. \tag{1.60a}$$

Combining the relations (1.58) and (1.60) one obtains:

$$-kx = \frac{d^2x}{dt^2};\ \frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \tag{1.60b}$$

By the denotation:

$$\omega_0^2 = \frac{k}{m} \tag{1.61}$$

one obtains:

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0. \tag{1.62}$$

Equation (1.62) is the differential equation of the non-damped/simple harmonic oscillator. It is a 2nd order homogeneous linear differential equation. The so called characteristic equation is:

$$r^2 + \omega_0^2 = 0 \Rightarrow r_{1,2} = \pm i\omega_0. \tag{1.63a}$$

The solution to the equation (1.62) is:

$$x = B_1 e^{i\omega_0 t} + B_2 e^{-i\omega_0 t}. \tag{1.63b}$$

Using Euler's formulas $e^{\pm ix} = \cos x \pm i \sin x$, one gets:

$$x = (B_1 + B_2)\cos\omega_0 t + i(B_1 - B_2)\sin\omega_0 t. \tag{1.63c}$$

We denote:

$$C_1 = B_1 + B_2 \;\; C_2 = i(B_1 - B_2)$$

and one has:

$$x = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t \tag{1.63d}$$

and substitute:

$$C_1 = A\sin\varphi; \qquad C_2 = A\cos\varphi,$$

it results:

$$x = A(\sin\varphi\cos\omega_0 t + \cos\varphi\sin\omega_0 t) \tag{1.63e}$$

and

$$x = A\sin(\omega_0 t + \varphi). \tag{1.63f}$$

This relation represents the law of the non-damped/simple harmonic oscillatory motion. Alternatively, with the formula $\sin\alpha = \cos(\alpha - \pi/2)$, the relation (1.63*f*) can be written as follows:

$$x = A\cos(\omega_0 t + \varphi_1)\ \ \varphi_1 = \varphi - \frac{\pi}{2}, \qquad (1.64)$$

where x is the displacement, A is the amplitude that equals the maximum displacement. ω_0 is the *angular frequency* of the solution, as it is determined only by its characteristics, (measured in radians per second). $\omega_0 t + \varphi$ is the phase, while φ – initial phase (at $t = 0$).

The period T_0 is the minimum time during which the material point stays in the same position, has the same speed, in terms of value and direction (therefore, the duration of a complete oscillation). We have $x(t) = x(t + T_0)$, $A\sin(\omega_0 t + \varphi) = A\sin[\omega_0(t + T_0) + \varphi]$, $\omega_0(t + T_0) + \varphi = \omega_0 t + \varphi + 2\pi$, $T_0 = 2\pi/\omega_0$.

Using relation (1.61) one obtains:

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}. \qquad (1.65)$$

The *frequency* (measured in hertz), is given by:

$$\nu_0 = \frac{1}{T_0}. \qquad (1.66)$$

We determine the speed and acceleration of the material point undergoing harmonic oscillations (oscillatory motion) as follows:

$$\mathrm{v} = \mathrm{v}_x = \frac{dx}{dt} = \omega_0 A\cos(\omega_0 t + \varphi), \qquad (1.67)$$

$$a = a_x = -\omega_0^2 A \sin(\omega_0 t + \varphi) = -\omega_0^2 x\,. \qquad (1.68)$$

In simple oscillations, acceleration is proportional and inversely directed to the displacement.

We determine kinetic, potential and total energy of the oscillator at any t time as follows:

$$T = \frac{m\mathrm{v}^2}{2} = \frac{1}{2} m\omega_0^2 A^2 \cos^2(\omega_0 t + \varphi)\,, \qquad (1.69)$$

$$U = \frac{kx^2}{2} = \frac{1}{2} m\omega_0^2 A^2 \sin^2(\omega_0 t + \varphi)\,, \qquad (1.70)$$

$$E = T + U = \frac{1}{2} m\omega_0^2 A^2 = \frac{1}{2} k A^2\,. \qquad (1.71)$$

The total energy is constant relative to time. Consequently, the simple harmonic oscillator is a conservative system.

The Complex Numbers and the Harmonic Oscillation

The displacement of the harmonic oscillatory motion is represented by the complex number:

$$x = Ae^{i(\omega_0 t + \varphi)},\; x = A[\cos(\omega_0 t + \varphi) + i\sin(\omega_0 t + \varphi)]. \qquad (1.72)$$

Under these conditions, the following convention is made: the displacement is given by the imaginary part of this complex.

The Vectorial Representation of the Harmonic Oscillation

Any harmonic oscillation can be graphically represented, by a rotating vector $\vec{A}$, called *phasor* (Fig. 1.9). The maximum displacement is given by the phasor's length and the phasor rotates in trigonometric (counterclockwise) direction at an angular frecquency ω_0. The phasor travels with velocity $2\pi A / T$, which is the maximum speed of the oscillator. So, the angle it makes with the *x*-axis gives the phase angle.

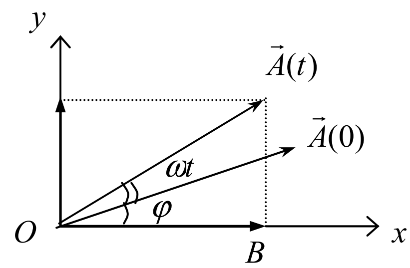


Fig. 1.9

At $t = 0$, the phase φ denotes the angle made by the phasor with axis Ox, while at any t time, we have $\omega_0 t + \varphi$. The projection B of the terminal point of the phasor on axis Ox (as well as the projection on Oy) describes a harmonic oscillation. The vector method (phasor) is mainly used in the study of the two parallel/orthogonal harmonic oscillations of equal frequency.

1.2.3. Particular Case of Simple/non-Damped Harmonic Oscillation. Simple Pendulum (Bob Pendulum).

Simple (mathematical) pendulum consists of a weight m attached to a massless inelastic wire (rigid rod). Because of an initial push, it will swing back and forth under the influence of gravity over its central point. We assume that the bob is a point mass and motion occurs in a 2 dimensional plane (Fig. 1.10). The harmonic oscillations of the pendulum occur for small angular displacements about the equilibrium point, namely $\alpha < 4°$ (small angle approximation). The motion is caused by the tangent component of $\vec{G}$, G_t:

$$G_t = -mg\sin\alpha \cong -mg\alpha\,, \qquad (1.74a)$$

with minus because the gravitational force on the bob causes a *decrease* in angle.

We apply the fundamental law of dynamics:

$$G_t = ma = m\frac{d\mathrm{v}}{dt}. \qquad (1.74b)$$

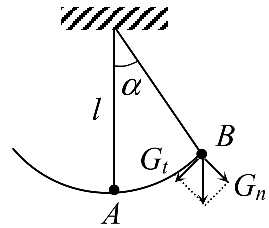


Fig. 1.10

We calculate linear acceleration a_t ($AB = s$):

$$s = \alpha l \Rightarrow \mathrm{v} = \frac{ds}{dt} = l\frac{d\alpha}{dt} \Rightarrow a_t = \frac{d\mathrm{v}}{dt} = l\frac{d^2\alpha}{dt^2}\,, \qquad (1.74\mathrm{c})$$

$$G_t = ml\frac{d^2\alpha}{dt^2}. \qquad (1.74\mathrm{d})$$

From the two expressions of G_t one gets:

$$-mg\alpha = ml\frac{d^2\alpha}{dt^2} \Rightarrow \frac{d^2\alpha}{dt^2} + \frac{g}{l}\alpha = 0. \qquad (1.74\mathrm{e})$$

Let us denote:

$$\frac{g}{l} = \omega_0^2 \Rightarrow \frac{d^2\alpha}{dt^2} + \omega_0^2\alpha = 0. \qquad (1.74\mathrm{f})$$

We get an equation identical with the differential equation of the simple/non-damped harmonic oscillator, (1.62), α being replaced by x. Therefore the solution to the new equation is:

$$\alpha = \alpha_0\sin(\omega_0 t + \varphi) \qquad (1.74g)$$

and the period is given by:

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}\,. \tag{1.75}$$

Apllications

According to M. Schuler (1923), a pendulum whose period exactly equals the orbital period of a hypothetical satellite orbiting just above the surface of the earth (about 84 minutes) will tend to remain pointing at the center of the earth when its support is suddenly displaced. This represents the basic principle of Schuler tuning that has to be included in the design of any inertial guidance system that will be operated near the earth, such as in ships and aircraft.

Because of the values of the gravitational acceleration g in the equation (1.75) the pendulum frequency is different at different places on earth. If we consider an accurate pendulum clock in Glasgow (g = 9.815 63 m/s^2) and take it to Cairo (g = 9.793 17 m/s^2), we must shorten the pendulum by 0.23%.

Double pendulum:

- in horology, a double pendulum represents a system of two simple pendulums on a common mounting which move in anti-phase.

- in mathematics (dynamical systems), a double pendulum is a pendulum that has another pendulum attached to its end. This system is a simple physical system that exhibits rich dynamic behavior. A set of coupled ordinary differential equations describe the motion of a double pendulum. Also, for a value of energy greater than a certain one its motion is chaotic.

A property of the double pendulum is that it undergoes chaotic motion, and shows a sensitive dependence on initial conditions. The image from Fig. 1.11 shows the amount of elapsed time before the pendulum "flips over", as a function of initial conditions *.

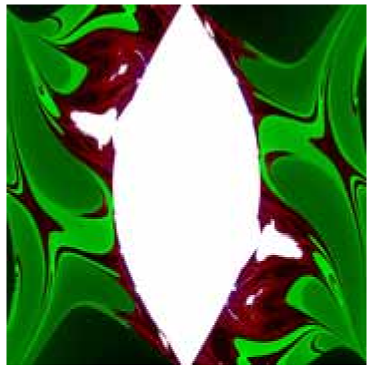


Fig. 1.11*

1.2.4. Damped Harmonic Oscillations

The harmonic oscillator is known as *damped* in the presence of a frictional force (damping) proportional to the velocity. The frequency of the oscillations is smaller than in the non-damped case, and the amplitude of the oscillations is not constant and decreases with time.

Damped oscillations are those oscillations with reduced amplitude due to the dissipation of energy, under the action of friction forces. The friction force acting upon the oscillator depends upon the given motional conditions. Let us consider the oscillator moving through fluid. In this case, at relatively low speed, the body is subject to a frictional force proportional to velocity, as in the Fig. 1.12:

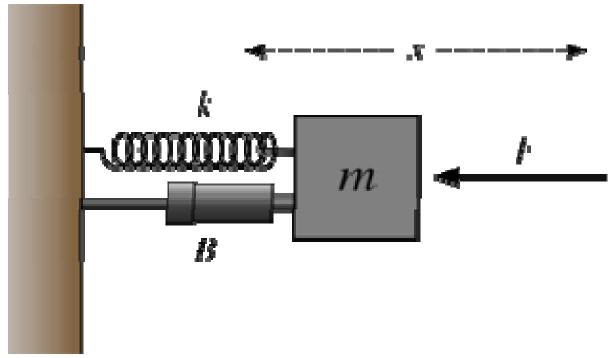


Fig. 1.12

A mass attached to a spring and a damper. (The *F* in the diagram denotes an external force, which this example does not include.)*

$$F_r = -\gamma \mathrm{v} = -\gamma \frac{dx}{dt}, \quad (1.76)$$

where γ is a positive coefficient that depends on the nature of the fluid and is called *damper constant*. The minus signs the fact that the frictional force opposes the body displacement and has its direction contrary to velocity. In the case of stringed instruments such as guitar or violin, *damping* represents the quieting or abrupt silencing of the strings after they have been sounded. The strings can be modeled as a continuum of infinitesimally small mass-spring-damper systems where the damping constant is much smaller than the resonant frequency, creating damped oscillations.

We consider that the displacement about the equilibrium point is small; consequently, the restoring force is elastic. According to the fundamental principle of dynamics, one gets:

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0 \Rightarrow \frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \omega_0^2 x = 0 \quad (1.77)$$

and we denote

$\frac{\gamma}{m}=2\delta$ and $\omega_0^2=\frac{k}{m}$, where δ is called *damping factor* and ω_0 is called the (undamped) natural frequency (angular frequency of the oscillator in the absence of friction) of the system. Both parameters represent angular frequencies and have for units of measure radians per second.

Using these denotations, the equation becomes:

$$\frac{d^2x}{dt^2}+2\delta\frac{dx}{dt}+\omega_0^2x=0. \tag{1.78}$$

Equation (1.78) is the differential equation of oscillations. It is a 2nd order homogeneous linear differential equation with constant factors.

To the equation (1.78) we attach the equation:

$$r^2+2\delta r+\omega_0^2=0. \tag{1.79}$$

The solutions to the equation (1.79) are:

$$r_{1,2}=-\delta\pm\sqrt{\delta^2-\omega_0^2}. \tag{1.80}$$

We distinguish two situations.

a) If the medium resistance is high, the factor γ gets a high value, so that $\delta^2-\omega_0^2>0$. In this case, solutions $r_{1,2}$ are real and the solution to the differential equation (1.78) becomes:

$$x=A_1e^{-(\delta-\sqrt{\delta^2-\omega_0^2})t}+A_2e^{-(\delta+\sqrt{\delta^2-\omega_0^2})t} \tag{1.81}$$

According to relation (1.81), displacement decreases exponentially with time. The motion is no longer periodical. The body, no longer in equilibrium, returns asymptotically to it, without exceeding it.

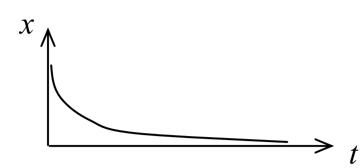


Fig.1.13

In this case, the motion is called *aperiodic* (Fig. 1.13). The system is said to be *over-damped*. An overdamped door-closer will take longer to close the door than a critically damped door closer.

When $\delta^2 - \omega_0^2 = 0$, δ is real and the system is *critically damped*. An example of critical damping is the door-closer of many hinged doors present in public buildings.

b) If the medium resistance is low, $\delta^2 - \omega_0^2 < 0$. In this case, $r_{1,2}$ are complex:

$$r_{1,2} = -\delta \pm i\sqrt{\omega_0^2 - \delta^2}\,. \tag{1.82}$$

We denote:

$$\omega = \sqrt{\omega_0^2 - \delta^2}\,. \tag{1.83}$$

The solution to (1.78) becomes:

$$x = A_1 e^{(-\delta + i\omega)t} + A_2 e^{(-\delta - i\omega)t} \tag{1.84}$$

and one obtains:

$$x = e^{-\delta t}(A_1 e^{i\omega t} + A_2 e^{-i\omega t})\,. \tag{1.85}$$

Using Euler's formula, one gets:

$$x = e^{-\delta t}[(A_1 + A_2)\cos \omega t + i(A_1 - A_2)\sin \omega t]\,. \tag{1.86}$$

By substitution:

$$A_1 + A_2 = A_0 \sin\varphi, \quad i(A_1 - A_2) = A_0 \cos\varphi\,. \tag{1.87}$$

The displacement becomes:

$$x = A_0 e^{-\delta t}(\sin\varphi\cos\omega t + \cos\varphi\sin\omega t) \tag{1.88}$$

and one has:

$$x = A_0 e^{-\delta t}\sin(\omega t + \varphi)\,. \tag{1.89}$$

In this case, the system is *under-damped*. In this case, the system oscillates at the damped frequency, which is a function of the natural frequency and the damping factor. Relation (1.89) represents the law of the damped oscillatory motion. The amplitude is:

$$A(t) = A_0 e^{-\delta t}. \tag{1.90}$$

The amplitude is not constant, but decreases exponentially (Fig. 1.14). With ω as the damped frequency the period T can be defined as the minimum time interval between two successive moves of the oscillator through the equilibrium point, in the same direction:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_0^2 - \delta^2} \tag{1.91}$$

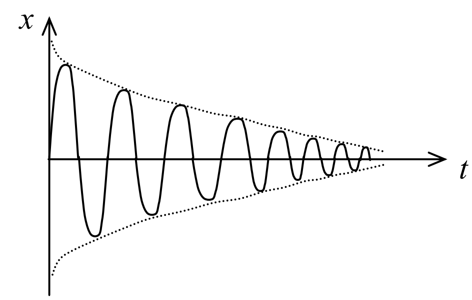


Fig. 1.14

and the period of the oscillations is bigger than in the non-damped case.

The damped oscillations are also determined by the following quantities:

i) *The damping ratio* (*decrement of damping*), β defined by the relation:

$$\beta = \ln \frac{A(t)}{A(t+T)} = \ln \frac{A_0 e^{-\delta t}}{A_0 e^{-\delta (t+T)}} = \ln e^{\delta T} = \delta T \tag{1.92}$$

and one gets:

$$\beta = \delta T. \tag{1.93}$$

The stronger the damping is, the higher β gets.

ii) *Equilibrium time of motion*, τ, is the time relative to which the amplitude decreases *e* times.

$$A(t+\tau)=\frac{A(t)}{e},\quad A_0e^{-\delta(t+\tau)}=\frac{A_0e^{-\delta(t)}}{e}\Rightarrow e^{-\delta\tau}=e^{-1}\Rightarrow\delta\tau=1$$

$$\tau=\frac{1}{\delta}. \tag{1.94}$$

The smaller τ is, the stronger the damping gets. We can develop the motion law regard to τ as below:

$$x=A_0e^{-t/\tau}\sin(\omega t+\varphi). \tag{1.95}$$

ii) *The quality* (*performance*) *factor*, Q is defined by the relation:

$$Q=\omega_0\tau. \tag{1.96}$$

This relation is valid for an oscillator which is not strongly damped. In aperiodic motion we have:

$$\delta\ge\omega_0\Rightarrow Q=\omega_0\tau=\frac{\omega_0}{\delta}\le 1. \tag{1.97}$$

In periodic motion:

$$\delta<\omega_0\Rightarrow Q=\frac{\omega_0}{\delta}>1. \tag{1.98}$$

The higher Q is, the more weakly the motion gets damped.

According to relation (1.89), oscillations stop after an infinite time. In reality, when amplitude becomes comparable to interatomic distance, we cannot speak of an oscillation of the whole body with such an amplitude. Thus, actually, the oscillations stop after a definite time.

In the damped oscillations, the energy that the oscillator needs in order to be displaced from the equilibrium point gradually turns into thermal energy due to friction. The damped oscillator is a dissipative system.

Analogy between the mechanical case and the electrical case, series RLC circuit:

- force $F\ (N)$ – voltage $U\ (V)$;
- speed v *(m/s)* – current intensity $I\ (A)$;
- displacement $x\ (m)$ – electric charge $q\ (C)$;
- damper constant $\gamma\ (kg/s)$ – resistance $R\ (\Omega)$;
- mass $m\ (kg)$ – inductance $L\ (H)$;
- spring constant $k\ (N/m)$ – inverse of capacitance (elastance) $\frac{1}{C}\ (1/F)$;
- $\frac{1}{2\pi}\sqrt{\frac{k}{m}}\ -\ \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$.

Analogy between the mechanical case and the electrical case, parallel RLC circuit:

- force $F\ (N)$– voltage $U\ (V)$;
- speed v *(m/s)* – $\frac{du}{dt}$;
- displacement $x\ (m)$ – voltage $U\ (V)$;
- damper constant $\gamma\ (kg/s)$ – conductance $\frac{1}{R}\ (\Omega^{-1})$;
- mass $m\ (kg)$ –capacitance $C\ (F)$;
- spring constant $k\ (N/m)$ – susceptance $\frac{1}{L}\ (1/H)$;
- $\frac{1}{2\pi}\sqrt{\frac{k}{m}}\ -\ \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$.

1.2.5 Driven Harmonic Oscillations. Resonance.

In order to prevent the decrease of amplitude, due to energy dissipation, external energy needs to be communicated to the oscillator. This can be achieved if a periodic external force acts upon the oscillator. In driven oscillations, the following forces act upon the oscillator: elastic force F_e, resistance force F_r, and a periodic force, $F_{ext} = F_0 \sin(\omega t + \varphi)$, where ω is the angular frequency of the external force. The oscillations are called driven, but they are also damped.

In the first moments, after the application of the external force, the oscillations are not stationary, in other words, they do not have a constant amplitude and pulsation. Meanwhile, the amplitude and pulsation vary according to a complicated law, the oscillations being under *transient conditions*. In time, the *steady-state conditions* are reached.

We determine the equation of motion and the law of motion:

$$F_e + F_r + F_{ext} = ma\,, \tag{1.99}$$

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = F_0 \sin(\omega t + \varphi)\,, \tag{1.100}$$

$$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \frac{k}{m}\,x = \frac{F_0}{m}\sin(\omega t + \varphi) \tag{1.101}$$

and we denote

$\frac{\gamma}{m} = 2\delta$ and $\omega_0^2 = \frac{k}{m}$, where δ is called *damping factor* and ω_0 is called the (undamped) natural frequency (angular frequency of the oscillator in the absence of friction) of the system. Both parameters represent angular frequencies and have for units of measure radians per second. We obtain:

$$\frac{d^2x}{dt^2}+2\delta\frac{dx}{dt}+\omega_0^2 x=\frac{F_0}{m}\sin(\omega t+\varphi). \tag{1.102}$$

This is the differential equation of the driven harmonic oscillations. It is an unhomogeneous 2nd order linear differential equation, with constant factors. The solution to this type of equation is the sum of the homogeneous equation solution and a particular solution, like the second member's, $x = x_{om} + x_{part}$.

The solution to the homogeneous equation is: $x_{om} = A_0 e^{-\delta t}\sin(\omega_1 t+\varphi)$.

The particular solution is: $x = A\sin(\omega t+\varphi)$.

In time, due to the damping, $x_{om} \to 0$ and, as a result, the solution to equation (1.102) will be:

$$x = x_{part} = x = A\sin(\omega t+\varphi). \tag{1.103}$$

Since the moment we can have such a solution (1.103), the oscillator is under steady-state conditions. *Under these conditions, the oscillations have a frequency equal to the frequency of the external force*.

Considering (1.103) a solution to the equation (1.102), we can determine the constants A and φ.

$$\frac{dx}{dt}=\omega A\cos(\omega t+\varphi), \tag{1.104}$$

$$\frac{d^2x}{dt^2}=-\omega^2 A\sin(\omega t+\varphi). \tag{1.105}$$

Introducing (1.103), (1.104) and (1.105) in (1.102) it follows:

$$(\omega_0^2-\omega^2)A\sin(\omega t+\varphi)+2\delta\omega A\cos(\omega t+\varphi)=\frac{F_0}{m}\sin(\omega t+\varphi). \tag{1.106}$$

By developing sinus and cosinus and equating the coefficients of $\sin\omega t$ and $\cos\omega t$ it follows:

$$A(\omega_0^2-\omega^2)\cos\varphi-2\delta\omega A\sin\varphi=\frac{F_0}{m}, \qquad (1.107)$$

$$A(\omega_0^2-\omega^2)\sin\varphi+2\delta\omega A\cos\varphi=0. \qquad (1.108)$$

From relation (1.108) one gets:

$$tg\varphi=\frac{2\delta\omega}{\omega^2-\omega_0^2}. \qquad (1.109)$$

From the multiplication of (1.107) by $\cos\varphi$ and (1.108) by $\sin\varphi$ and sum of the new relations, one obtains:

$$A=\frac{F_0\cos\varphi}{m(\omega_0^2-\omega^2)}. \qquad (1.110)$$

We develop $\cos\varphi$:

$$\cos\varphi=\frac{1}{\sqrt{1+tg^2\varphi}}=\frac{\pm(\omega^2-\omega_0^2)}{\sqrt{(\omega^2-\omega_0^2)^2+4\delta^2\omega^2}}. \qquad (1.111)$$

Combining (1.110) and (1.111) one obtains:

$$A=\frac{F_0}{m\sqrt{(\omega_0^2-\omega^2)^2+4\delta^2\omega^2}}. \qquad (1.112)$$

According to relations (1.109) and (1.111), the amplitude, A and the initial phase, φ of the driven oscillations depend on pulsation ω of the external force. The angular frequency value/modulus of the external force with maximum amplitude is called *angular resonant frequency* ω_r. The phenomenon of developing a driven oscillation of maximum amplitude when $\omega = \omega_r$, is called *resonance*. The angular resonant frequency is obtained from the maximum condition:

$$\frac{dA}{d\omega} = 0 \Rightarrow 4\omega(\omega^2 - \omega_0^2 + 2\delta^2) = 0 \tag{1.113}$$

and we get

$$\omega_r = \sqrt{\omega_0^2 - 2\delta^2}\ . \tag{1.114}$$

Embedding (1.114) in (1.112) we determine the value of the amplitude relative to resonance:

$$A_r = \frac{F_0}{2\delta m\sqrt{\omega_0^2 - \delta^2}}\,. \tag{1.115}$$

The amplitude with regard to resonance is as higher as the damping factor/coefficient, δ is lower. If the medium resistance is null, $\delta = 0$, $\omega_r = \omega$ and $A_r \to \infty$. This situation does not actually occur, as the medium resistance always interferes. The curves, as a result of the graph representation of the amplitude relative to the angular frequency of the external force, are called *resonance curves*. (Fig. 1.15). An example of resonance in Fig. 1.16.

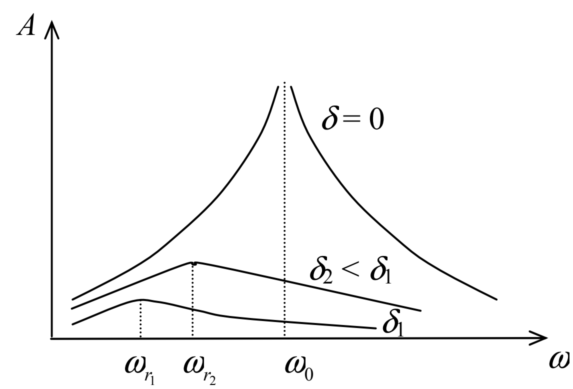


Fig. 1.15

Fig. 1.16

The Tacoma Narrows Bridge (shown twisting) in Washington collapsed spectacularly, under moderate wind, in part because of resonance*.

In physics, *resonance* is the tendency of a system to absorb more energy when the frequency of its oscillations matches the system's natural frequency of vibration (its *resonant frequency*) than it does at other frequencies. Examples of resonance: the acoustic resonances of musical instruments, the tidal resonance, orbital resonance as exemplified by some moons of the solar system's gas giants, the resonance of the basilar membrane in the biological transduction of auditory input, and resonance in electronic circuits.

1) *Acoustic resonances*. Strings under tension (lutes, harps, guitars, pianos) have resonant frequencies which are directly connected to the mass, length, and tension of the string. The wavelength that corresponds to the first resonance on the string is equal to twice the length of the string. For higher resonances correspond wavelengths that are integer divisions of the fundamental wavelength.

2) *Tidal resonance*. *Tidal resonance* (oceanography) occurs when the time it takes for a large wave to travel from the mouth of the bay to the opposite end, then

reflect and travel back to the mouth of the bay, equals the time from one high tide to the next.

3) *Orbital resonance* appears in the case of some moons of the solar system's gas giants. When two bodies have periods of revolution that are a simple integer ratio of each other a mean motion orbital resonance appears. Depending on the conditions, the orbit can be either stabilized or destabilized. When the two bodies move in a synchronized fashion and they never closely approach *stabilization* occurs. There are four gas giants in our solar system's: Jupiter, Saturn, Uranus, and Neptune (Fig. 1.17). Uranus and Neptune are a separate subclass of giant planets, 'ice giants', or 'Uranian planets', because they are mostly composed of ice, rock and gas, unlike the "traditional" gas giants Jupiter or Saturn. They share the same qualities of the lack of the solid surface; their differences stem from the fact that their proportion of hydrogen and helium is lower, because they are situated at greater distance from the Sun.

Fig. 1.17

From top: Neptune, Uranus, Saturn, and Jupiter (sizes not to scale)*.

4) *Resonance of the basilar membrane in the biological transduction of auditory input*. The membrane is tapered and it is stiffer at one end than at the other. Because of this, there is a sound input (frequency) that makes to vibrate a particular location of the membrane more than other locations due to the physical property of resonance. Georg von Békésy (Nobel Prize) showed in experiments that high frequencies lead to maximum vibrations at the basal end of the cochlear coil (narrow membrane), and low frequencies determine maximum vibrations at the apical end of the cochlear coil (wide membrane).

5) *Resonance in electronic circuits*. In an electrical circuit, resonance appears at a particular frequency when the inductive reactance equals the capacitive reactance. This determines electrical energy to oscillate between the magnetic field of the inductor and the electric field of the capacitor. An analogy is represented by the mechanical pendulum. At resonance, the series impedance of the two elements is at a minimum and the parallel impedance is a maximum. Resonance has applications in tuning and filtering, because resonance appears at a particular frequency for given values of inductance and capacitance. Resonance can produce unwanted sustained and transient oscillations in electrical circuits that may cause noise, signal distortion, and damage to circuit elements.[*]

1.2.6. Mechanical Oscillations

a) Two Parallel Harmonic Oscillatory Motions of Equal Frequency

Let us consider a material point undergoing two harmonic oscillatory motions of equal frequency along axis *Ox*:

$$x_1 = A_1 \sin(\omega t + \varphi_1), \tag{1.116}$$

$$x_2 = A_2 \sin(\omega t + \varphi_2). \tag{1.117}$$

The displacement of the resulted motion is given by:

$$\begin{aligned} x = x_1 + x_2 &= A_1 \sin \omega t \cos \varphi_1 + A_1 \cos \omega t \sin \varphi_1 + \\ &+ A_2 \sin \omega t \cos \varphi_2 + A_2 \cos \omega t \sin \varphi_2 = \\ &= (A_1 \cos \varphi_1 + A_2 \cos \varphi_2) \sin \omega t + \\ &+ (A_1 \sin \varphi_1 + A_2 \sin \varphi_2) \cos \omega t \end{aligned}. \tag{1.118}$$

We denote:

$$B_1 = A_1 \cos \varphi_1 + A_2 \cos \varphi_2,\ B_2 = A_1 \sin \varphi_1 + A_2 \sin \varphi_2. \tag{1.119}$$

One gets:

$$x = B_1 \sin \omega t + B_2 \cos \omega t. \tag{1.120}$$

By substitution:

$$B_1 = A \cos \varphi;\ B_2 = A \sin \varphi \tag{1.121}$$

and the displacement becomes:

$$x = A \sin(\omega t + \varphi). \tag{1.122}$$

The resulted motion is a harmonic oscillatory motion as well. Let us determine amplitude A and the initial phase φ of the resulted motion:

$$A\cos\varphi = A_1 \cos\varphi_1 + A_2 \cos\varphi_2, \tag{1.123}$$

$$A\sin\varphi = A_1 \sin\varphi_1 + A_2 \sin\varphi_2. \tag{1.124}$$

By dividing these relations, it follows that:

$$tg\varphi = \frac{A_1 \sin\varphi_1 + A_2 \sin\varphi_2}{A_1 \cos\varphi_1 + A_2 \cos\varphi_2}. \tag{1.125}$$

Squaring and summing relations (1.123) and (1.124), we get:

$$A^2 = A_1^2 + A_2^2 + 2A_1 A_2 \cos(\varphi_2 - \varphi_1). \tag{1.126}$$

The vector method leads to the same result. The vector representation (Fig. 1.18) is useful mainly in the composition study of more oscillations, of similar frequency and direction. In this case, the polygon rule applies: a vector polygon is drawn; the vector that closes up the polygon outline represents the resulted oscillation (Fig. 1.19).

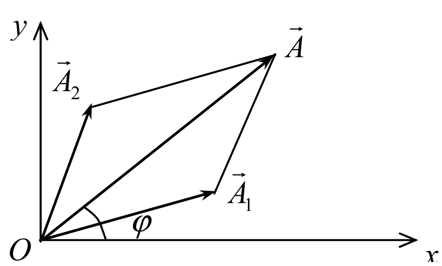


Fig. 1.18

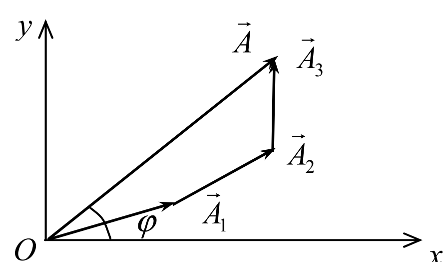


Fig. 1.19

Particular Cases

1) If $\varphi_2 - \varphi_1 = 2k\pi$, from relation (1.126) it follows

$$A = A_1 + A_2 \tag{1.127}$$

The resulted amplitude equals the sum of the component amplitudes. The two oscillations are considered *in phase* (Fig. 1.20).

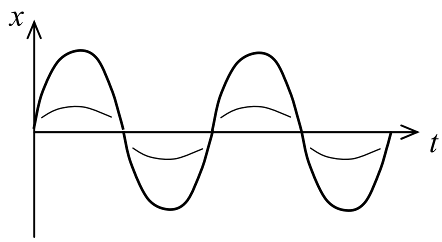


Fig. 1.20

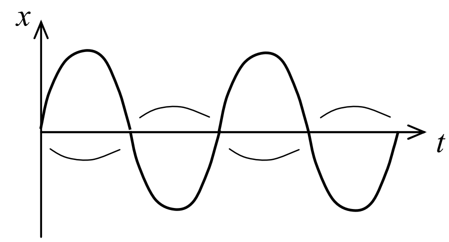


Fig. 1.21

2) If $\varphi_2 - \varphi_1 = 2(k+1)\pi$, from relation (1.126) one gets:

$$A = A_1 - A_2 . \tag{1.128}$$

The oscillations, in this case, are in phase opposition (Fig. 1.21).

3) If $\varphi_2 - \varphi_1 = 2(k+1)\pi / 2$, from relation (1.126) it results:

$$A^2 = A_1^2 + A_2^2 . \tag{1.129}$$

The two oscillations are in a square relationship.

b) Two 0rthogonal Harmonic oscillatory Motions of Same Frequency

Let us consider that a material point simultaneously undergoes two harmonic oscillations, one along *Ox*, the other along *Oy:*

$$x = A \sin \omega t , \tag{1.130}$$

$$y = B \sin(\omega t + \varphi) . \tag{1.131}$$

In order to obtain the trajectory equation, we leave out the time from the relations.

As a result, relation (1.130) becomes:

$$\sin \omega t = \frac{x}{A} \Rightarrow \cos \omega t = \sqrt{1-\sin^2 \omega t} = \sqrt{1-\frac{x^2}{A^2}} \tag{1.132}$$

Combining (1.131) and (1.132) one obtains:

$$y = B \sin \omega t \cos \varphi + B \cos \omega t \sin \varphi,$$

$$y = B \frac{x}{A} \cos \varphi + B \sqrt{1-\frac{x^2}{A^2}} \sin \varphi,$$

$$y - B \frac{x}{A} \cos \varphi = B \sqrt{1-\frac{x^2}{A^2}} \sin \varphi. \tag{1.133}$$

By squaring this relation, one gets:

$$\frac{y^2}{B^2} + \frac{x^2}{A^2} - \frac{2xy}{AB} \cos \varphi = \sin^2 \varphi. \tag{1.134}$$

The trajectory is an ellipse. The consecutive motion is an elliptical periodical motion (Fig. 1.22).

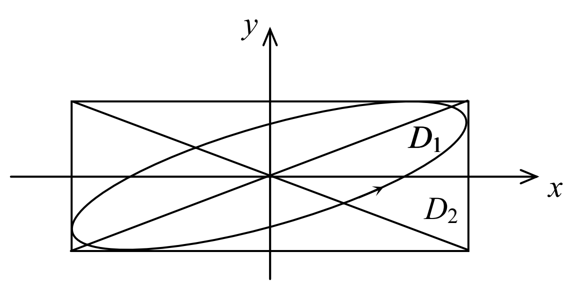


Fig. 1.22

Particular Cases

1) If $\varphi_2 - \varphi_1 = 2k\pi$, from relation (1.134) it follows:

$$\frac{y^2}{B^2}+\frac{x^2}{A^2}-\frac{2xy}{AB}=0 \Rightarrow \frac{x}{A}=\frac{y}{B}. \quad (1.135)$$

The consecutive motion is a harmonic oscillatory motion along line D_1 that represents the first diagonal from the amplitude rectangular.

2) If $\varphi_2-\varphi_1=2(k+1)\pi$, from relation (1.134) one gets:

$$\frac{x}{A}=-\frac{y}{B}. \quad (1.136)$$

The consecutive motion is a harmonic oscillatory motion along line D_2.

3) If $\varphi_2-\varphi_1=2(k+1)\pi/2$, from relation (1.134) it results:

$$\frac{y^2}{B^2}+\frac{x^2}{A^2}=1. \quad (1.137)$$

The motion is periodical on an ellipse around an axis. In particular, if $A=B$, the ellipse becomes a circle:

$$x^2+y^2=A^2. \quad (1.138)$$

Reciprocally, any circular periodical motion can be decomposed in two cross harmonic oscillations, of similar ν and A, having the phase difference $2(k+1)\pi/2$.

If the frequencies are different, the consecutive trajectory has a more complicated form. If the frequency ratio is a rational number, $\nu_1/\nu_2=n_1/n_2$, $n_1,n_2\in N$ what we get are closed trajectories, called Lissajous figures.

Chapter 2

Waves

2.1 Definition. Classification. Physical Description of a Wave. Wave Equation. Propagation of Waves. Energy of Waves. Wave Intensity.

2.1.1 Definition. Classification.

A *wave* represents a disturbance that propagates through space, often transferring energy. A mechanical wave exists in a medium but waves of an electromagnetic radiation, and probably gravitational radiation can travel through vacuum. Waves travel and transfer energy from one point to another, without any of the particles of the medium being displaced permanently (there is no associated mass transport) but there are oscillations around fixed positions.

Waves are characterized by *crests* (highs, maximums) and *troughs* (lows, minimums), either perpendicular (in the case of transverse waves) or parallel (in the case of longitudinal waves) to wave motion.

A wave can be described by the *wave function* as:

$$\Psi = f(x, y, z, t) = \psi(\vec{r}, t). \tag{2.1}$$

The wave function can be a *vector* (a displacement in mechanics, magnetic field $\vec{H}$, electric field intensity $\vec{E}$) or a scalar quantity (electrical potential difference, pressure).

The *wave front* is the locus (a line or surface in an electromagnetic wave) of the points that have the same phase. We denote:

$$\Psi(x, y, z, t) = S = const. \tag{2.2}$$

We have *plane waves* and *spherical waves*. Characteristic of a *plane wave*: has a constant-frequency and the wave fronts (surfaces of constant phase) are infinite parallel planes normal to the phase velocity vector. The waves that are approximately plane waves in a localized region of space are also called plane waves. For example, a localized source such as an antenna produces a field that is approximately a plane wave in its far-field region (the region beyond approximately 10 wavelengths from the antenna).

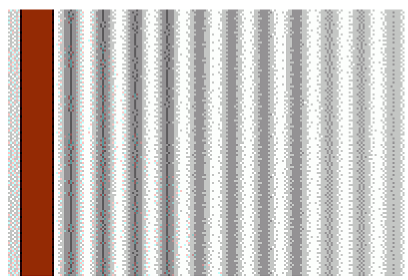


Fig. 1.23

Plane waves*

A *spherical wave* is the wave that has spherical wave front. For example, an antenna produces a field that is approximately a spherical wave in region less than the far-field region (the near-field region).

Spherical waves are the wave function given by:

$$\Psi = \Psi(x, y, x, t) = \Psi(r, t), \tag{2.3}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and:

$$\Psi = \frac{1}{r} F(r - \mathrm{v}t) \tag{2.4}$$

Also, we have *surface wave* which is a wave that is guided along the interface between two different media for a mechanical wave. Examples: the waves at the surface of water and air, ocean surface waves, or ripples in the sand on the interface of water or wind. Another example is internal waves, waves that are transmitted along the interface of two water masses of different densities. Surface waves are encountered in seismology. There is an analogy between surface waves and water waves. Surface waves travel over the Earth's surface. Their velocity is smaller than the velocity of body waves. Their low frequency determines them to be more likely than body waves to stimulate resonance in buildings. They are therefore the most destructive type of seismic wave. There are two types of surface waves: Rayleigh waves and Love waves.

Plane waves are described by $\Psi(x, y, t)$:

$$\Psi = F(\vec{r} \cdot \vec{k} - \mathrm{v}t), \tag{2.5}$$

where

$$\vec{r} = x\vec{i} + y\vec{j}\,. \tag{2.6}$$

Circular waves are described by $\Psi(r,t)$ with $r = \sqrt{x^2 + y^2}$ and:

$$\Psi = \frac{1}{\sqrt{r}} F(r - \mathrm{v}t) \tag{2.7}$$

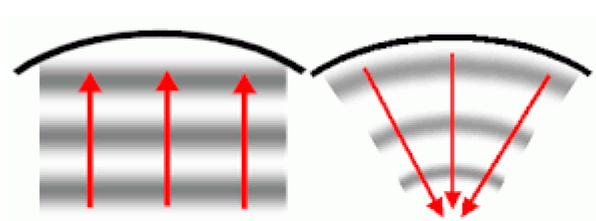


Fig. 1.24

Plane waves and circular waves*

Examples of Waves[*]:

1. Ocean surface waves. They represents perturbations that propagate through water (surfing and tsunami). *Ocean surface waves* are surface waves which appear at the surface of an ocean. They are waves guided along the interface between water and air. The wind transfers a part of its energy into the water, the water gains energy from the wind because of the friction between the wind and the water. The surface particles have an elliptical motion, which is a combination of longitudinal (back and forth) and transverse (up and down) wave motions.

Fig. 1.25

Breaking waves at La Jolla[*]

Individual "freak waves" (also "rogue waves", "monster waves" and "king waves") can occur in the ocean, often as high as 30 metres. Such waves are different by tides and tsunamis.

There are three main types of waves that are identified by surfers: plunging waves ("dumpers"), spilling waves and surging waves. Their properties make them more or less suitable for surfing and present different dangers.

2. Electromagnetic radiation: radio waves, microwaves, infrared rays, visible light, ultraviolet rays, x-rays, and gamma rays. In this case, propagation is possible

without a medium, through vacuum. These electromagnetic waves travel at 299 792 458 m/s in a vacuum.

In the case of *microwaves* the wavelengths is longer than those of infrared light, but relatively short for radio waves. Microwaves have wavelengths approximately in the range of 30 cm (frequency = 1 GHz) to 1 mm (300 GHz).

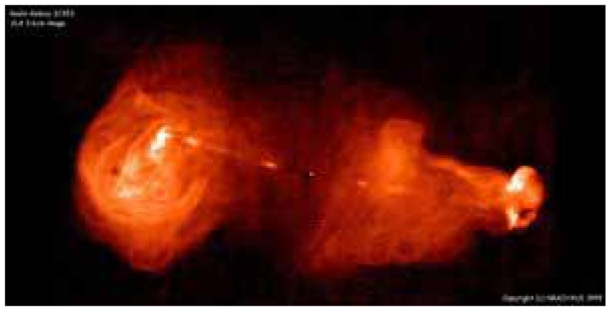


Fig. 1.26

Microwave image of 3C353 galaxy at 8.4 GHz (36 mm)*.

Infrared (IR) radiation is electromagnetic radiation of a wavelength longer than that of visible light, but shorter than that of microwave radiation. The name means "below red" (from the Latin *infra*, "below"), red being the color of visible light of longest wavelength. Infrared radiation spans three orders of magnitude and has wavelengths between approximately 750 nm and 1 mm.

The *visible spectrum* (*optical spectrum*) is the region of the electromagnetic spectrum that is visible to the human eye. Electromagnetic radiation in this range of wavelengths is called *visible light* or simply light. There are no fixed bounds to the visible spectrum, because a typical human eye can respond to wavelengths from 400 to 700 nm, although wavelengths from 380 to 780 nm can be perceived. A light-adapted eye has its maximum sensitivity at around 555 nm, in the green region of the optical spectrum. The spectrum does not contain all the colors that the human eyes and brain can distinguish, for example brown and pink are absent.

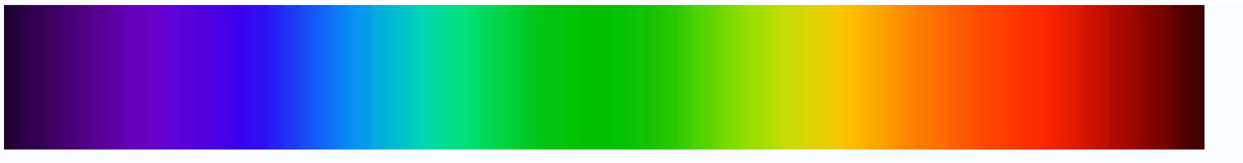


Fig. 1.27

Ultraviolet (UV) radiation is electromagnetic radiation of a wavelength shorter than that of the visible region, but longer than that of soft X-rays. There is *near UV* (380–200 nm wavelength), *far* or *vacuum UV* (200–10 nm FUV or VUV), and *extreme UV* (1–31 nm EUV or XUV).

X-rays have a wavelength in the range of 10 to 0.1 nanometers, corresponding to frequencies in the range 30 to 3000 PHz. X-rays are primarily used for diagnostic medical imaging and crystallography. Because X-rays are a form of ionizing radiation they can be dangerous.

Gamma rays (γ) are electromagnetic radiation produced by radioactive decay or other nuclear or subatomic processes such as electron-positron annihilation. They are begin at an energy of 10 keV, a frequency of 2.42 EHz, or a wavelength of 124 pm, although electromagnetic radiation from around 10 keV to several hundred keV is also referred to as hard X rays. Gamma rays and X rays of the same energy do not present physical difference. Gamma rays are distinguished from X rays by their origin. Gamma rays are a form of ionizing radiation.

3. Sound is a mechanical wave that propagates through air, liquid or solids. Its frequency can be detected by the auditory system. Similar are seismic waves in earthquakes, S, P and L kinds.

4. Gravitational waves, which are fluctuations in the gravitational field predicted by General Relativity. These waves are nonlinear.

2.1.1.1 Transverse and Longitudinal Waves

Other classification of the waves includes *transverse waves* and *longitudinal waves*. This classification is made in function of the position of the vibrations and the direction of the propagation of the wave.

Transverse waves are the waves with vibrations perpendicular to the direction of the propagation of the wave. Some examples: waves on a string, seismic waves and electromagnetic waves.

Longitudinal waves are the waves with vibrations parallel to the direction of the propagation of the wave. Some examples include most sound waves, ripples in water, and certain types of waves from earthquakes, where the particle motion is in the direction of travel.

In seismology transverse waves are called S (for "secondary") waves as they arrive later than the P ("primary") waves from an earthquake. The absence of transverse waves traveling through the earth's core demonstrates that it is liquid.

Also, transverse waves are connected to the *curl* operator and are governed by a vector wave equation. The longitudinal waves are connected to the *div* operator and are governed by a scalar wave equation. A longitudinal wave represents compressions moving through a plane. In this case the energy from the wave is transmitted as mechanical energy. An example is a slinky which was pushed forward and backwards, compressing and extending it as the motion of the wave was transmitted. Opposite, light is composed of transverse waves (electric component $\vec{E}$ and magnetic component $\vec{H}$).

Examples of combination of transverse and longitudinal waves are the Ripples on the surface of a pond. Therefore, the points on the surface follow elliptical paths*.

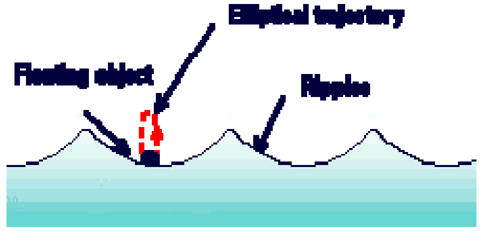


Fig. 1.28*

For an object that bobs up and down on a ripple in a pond, the trajectory is elliptical because ripples are not simple transverse sinusoidal waves.

2.1.1.2 Physical Description of a Wave

There are a number of standard variables that can be used for describing the waves. These standard variables are: amplitude, frequency, wavelength and period. The amplitude of a wave represents the magnitude of the maximum disturbance in the medium during one wave cycle. The amplitude is measured in units depending on the type of wave. For examples, waves on a string have an amplitude expressed in meters, sound waves as pressure (pascals) and electromagnetic waves have the amplitude expressed in units of the electric field (volts/meter). If the amplitude is constant we have the case of the *continuous wave* or the amplitude can vary with time and (or) position. The *envelope* of the wave is the form of the variation of amplitude.

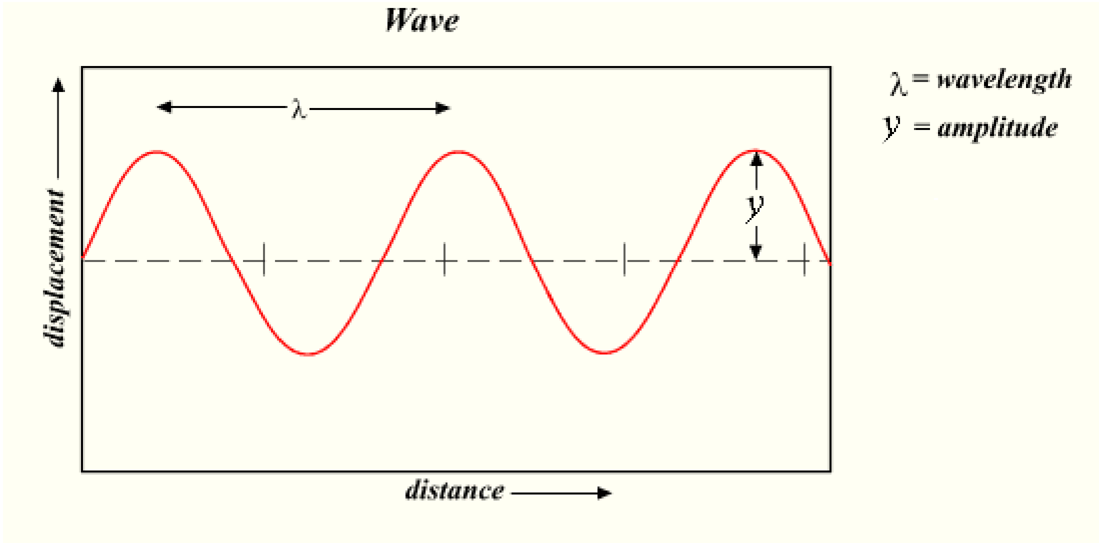


Fig. 1.29*

The highest point of a wave is called crest (maximum)and the trough is the lowest point (minimum) of a wave. The distance between two crests or two troughs that are beside each other represents the wavelength λ. For electromagnetic radiation, it is usually measured in nanometres.

The period of the wave T represents the time needed for one complete cycle for an oscillation of a wave. The frequency ν is the inverse of period or how many periods per unit time (for example one second). Frequency is measured in hertz. Also, we have the relation:

$$\nu = \frac{1}{T}. \tag{2.8}$$

If we want to express mathematically the waves we can use the *angular frequency* (ω, radians/second) which is related to the frequency ν by:

$$\omega = 2\pi\nu. \tag{2.9}$$

2.1.1.3 Travelling Waves

Standing waves are the waves that remain in one place. Examples: vibrations on a violin string, electromagnetic standing waves. Waves that are in movement are called *travelling waves*. They present a disturbance that varies both with time t and distance x. This can be expressed mathematically as:

$$\Psi(x,t) = A(x,t)\sin(\omega t - kx + \varphi), \tag{2.10}$$

where $A(x,t)$ is the amplitude envelope of the wave, k is the *wave number* and φ is the phase. The velocity v of this wave is given by:

$$\mathrm{v} = \frac{\omega}{\mathrm{k}} = \lambda\nu, \tag{2.11}$$

where λ is the *wavelength* of the wave.

The *phase velocity* of a wave represents the rate at which the phase of the wave propagates in space. In some cases the phase velocity of electromagnetic radiation can exceed the speed of light in a vacuum. No superluminal information or energy transfer are indicated. The phase velocity can be different by the *group velocity* of the wave. The *group velocity* represents the rate that changes in amplitude (the *envelope* of the wave) will propagate.

Also, we can write in the case when the propagation is towards the $0x$ axis:

$$\Psi(x,t) = f(t - \frac{x}{\mathrm{v}}) = F(x - \mathrm{vt}) = A\sin\omega(t - \frac{x}{\mathrm{v}}). \tag{2.11.a}$$

The quantity $\omega(t - \frac{x}{\mathrm{v}})$ represents an angle. For Ψ is available the periodicity in time with $T = \frac{2\pi}{\omega}$ and, also, the periodicity in space with $\lambda = \mathrm{v}T$. The expression for Ψ can be written:

$$\Psi(x,t) = A\sin\omega(t - \frac{x}{\mathrm{v}}) = A\sin 2\pi(\frac{t}{T} - \frac{x}{\lambda}) = A\sin(\omega t - kx), \quad (2.11.b)$$

where:

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{\mathrm{v}T} = \frac{\omega}{\mathrm{v}} \qquad (2.11.c)$$

is the *wave number* and is the absolute value of the *wave vector* $\vec{k}$

$$\vec{k} = k\vec{n} = \frac{2\pi}{\lambda}\vec{n} = \frac{\omega}{\mathrm{v}}\vec{n}. \qquad (2.11.d)$$

In the case of the propagation of the wave in space in the direction of the wave vector $\vec{k}$ we have:

$$\Psi(x,t) = A\sin(\omega t - k\vec{r}) = \mathrm{Im}\{Ae^{i(\omega t - k\vec{r})}\}, \qquad (2.11.e)$$

where

$$\varphi = \omega t - k\vec{r} \qquad (2.11.f)$$

is the phase of the wave.

The *group velocity* of a wave is the velocity with which the *envelope* of the wave propagate through space. The group velocity is defined by the equation:

$$\mathrm{v_g} = \frac{\partial\omega}{\partial \mathrm{k}}. \qquad (2.12.a)$$

About the group velocity we can say that is the velocity at which energy or information is conveyed along a wave. There is a direct connection between the group velocity and the *dispersion*. *Dispersion* is the phenomenon that causes the separation of a wave into components of varying frequency (wavelength). If ω is directly proportional to k, then the group velocity is exactly equal to the phase velocity. In the propagation of signals through optical fibers and in the design of short pulse lasers the "group velocity dispersion" is playing an important role.

2.1.2 Wave Equation

The *wave equation* is a partial differential equation. The wave equation describes many types of waves such as the sound waves, light waves and water waves. It is important in acoustics, electromagnetics, and fluid dynamics.

Also, the wave equation is an example of a hyperbolic partial differential equation. The wave equation (simplest form) refers to a scalar quantity Ψ, the wave function that satisfies:

$$\frac{\partial^2\Psi}{\partial t^2} = \mathrm{v}^2\nabla^2\Psi, \qquad (2.12.b)$$

where v is the propagation speed of the wave. Also, for the case of $\vec{r} = \vec{r}(x, y, z)$ one gets:

$$\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi}{\partial y^2} - \frac{\partial^2\Psi}{\partial z^2} - \frac{1}{\mathrm{v}^2}\frac{\partial^2\Psi}{\partial t^2} = 0. \qquad (2.12.c)$$

For example, for a sound wave in air at 20°C the speed is about 343 m/s. The speed can have a wide variation, in the case of the vibration of a string depending upon the linear density of the string and the tension on it. For example, for a spiral spring it can be as slow as a meter per second. When we have dispersion, the differential equations for waves allows the speed of wave propagation to vary with the frequency of the wave.

In this case, we have to replace v in (2.12.b) by the phase velocity:

$$\mathrm{v} = \frac{\omega}{\mathrm{k}}. \qquad (2.12.d)$$

2.1.3 Propagation of Waves

2.1.3.1 Propagation through Solids

1) Longitudinal Waves

In solids, there are two kinds of sound waves, *pressure waves* and *shear waves*. A non-zero stiffness both for volumetric and shear deformations is present in a solid.

In a solid there are conditions for generating sound waves with different velocities dependent on the deformation mode.

We have two cases:

a) the propagation of longitudinal waves (pressure waves) through a solid rod when the length of the solid rod is finite (a bounded solid rod) or with thickness much smaller than the wavelength, and

b) the propagation of transverse waves (shear waves) when the length of the solid rod is infinite (boundless) or with lateral dimensions much larger than the wavelength.

a) We consider a solid rod and the length of the solid rod is finite (a bounded solid rod). We denote the cross section of the rod with S, the linear density with ρ and the *modulus of elasticity* or *Young's modulus* with E. A vibration in a rod is a wave. Tension is a reaction force applied by a stretched rod (or a similar object) on the objects which stretch it. The direction of the force of tension is parallel to the rod, towards the rod. The magnitude of the force of tension has an increasing with the amount of stretching. In the case of a small stretching, the force is given by Hooke's law. The Hooke's law is given by:

$$\sigma(x,t)=\frac{1}{S}F(x,t)=E\varepsilon(x,t)=E\frac{\partial\Psi}{\partial x}, \tag{2.13}$$

where the *extension* (*strain*) is linearly proportional to its *tensile stress*, σ by a constant factor, the modulus of elasticity E. Also, one has:

$$\frac{1}{S}F(x,t)=E\frac{\Delta l}{l_0}=E\frac{l-l_0}{l_0} \tag{2.14}$$

and

$$\varepsilon=\frac{l-l_0}{l_0}=\frac{\partial\Psi}{\partial x}. \tag{2.15}$$

Also, we can write for the infinitesimal displacement dx with $dm=\rho S dx$ the resultant force:

$$dF=\frac{\partial\sigma}{\partial x}dxS. \tag{2.16}$$

From the second law of dynamics $F=ma$ one gets:

$$dF=dm\frac{\partial^2\Psi}{\partial t^2}=\frac{\partial\sigma}{\partial x}dxS \tag{2.17}$$

and

$$\rho S dx\frac{\partial^2\Psi}{\partial t^2}=\frac{\partial\sigma}{\partial x}dxS \tag{2.18}$$

and it results:

$$\rho\frac{\partial^2\Psi}{\partial t^2}=\frac{\partial\sigma}{\partial x}. \tag{2.19}$$

For the rod one has:

$$\sigma(x,t)=E\varepsilon(x,t)=E\frac{\partial\Psi}{\partial x} \tag{2.20}$$

and

$$\rho\frac{\partial^2\Psi}{\partial t^2}=\frac{\partial\sigma}{\partial x}=E\frac{\partial^2\Psi}{\partial x^2} \tag{2.21}$$

and it follows:

$$\frac{\partial^2\Psi}{\partial x^2}-\frac{\rho}{E}\frac{\partial^2\Psi}{\partial t^2}=0. \tag{2.22}$$

If we compare this relation to the wave equation:

$$\frac{\partial^2\Psi}{\partial x^2}-\frac{1}{\mathrm{v}^2}\frac{\partial^2\Psi}{\partial t^2}=0, \tag{2.23}$$

we obtain the speed of a wave traveling (the speed of sound) along a rod:

$$\mathrm{v}_l=\sqrt{\frac{E}{\rho}}. \tag{2.24}$$

The speed is directly proportional to the square root of the modulus of elasticity E over the linear density ρ. Thus in steel the speed of sound is approximately 5100 $\mathrm{m\cdot s^{-1}}$.

b) In the case when the length of the solid rod is infinite (boundless) or with lateral dimensions much larger than the wavelength the Hooke's law is given by:

$$\sigma(x,t)=E^{'}\varepsilon(x,t)=\frac{E(1-\mu)}{(1+\mu)(1-2\mu)}\varepsilon, \tag{2.25}$$

where μ is the *Poisson's coefficient* and $E^{'}\cong 1{,}35E$ is the *plane wave modulus* and $\dfrac{(1-\mu)}{(1+\mu)(1-2\mu)}$ is the *Poisson's ratio*. The speed of the longitudinal waves is given by:

$$\mathrm{v}_l^{'}=\sqrt{\frac{E^{'}}{\rho}}=\sqrt{\frac{E(1-\mu)}{(1+\mu)(1-2\mu)\rho}}>\mathrm{v}_l. \tag{2.26}$$

The sound velocity is higher than in the first case.

2) Transverse Waves

In the case of the transverse waves the resultant force that is acting upon the dm is perpendicular to the direction of propagation and we have the connection to the *lateral shear* given by:

$$dF = \frac{\partial \tau}{\partial x} dxS. \qquad (2.27)$$

Shear appears at the cutting of the sheet iron, at the stress of the rivet. Such deformations are homogeneous. The second law of dynamics is:

$$dm \frac{\partial^2 \Psi}{\partial t^2} = \frac{\partial \tau}{\partial x} dxS \qquad (2.28)$$

and one gets:

$$\rho \frac{\partial^2 \Psi}{\partial t^2} = \frac{\partial \tau}{\partial x}. \qquad (2.29)$$

From the Hooke's law it results:

$$\tau = G\gamma = G \frac{\partial \Psi}{\partial x}, \qquad (2.30)$$

where $\tau = F / S$ is the *shear stress*, G is the *shear modulus* or *modulus of rigidity* and γ is the *shear strain*. The shear modulus is a quantity used for measuring the strength of materials. The shear modulus describes the material's response to shearing strains. All of them arise in the generalized Hooke's law. The shear modulus influences the value of the speed of sound and also controls it.

One gets:

$$\rho \frac{\partial^2 \Psi}{\partial t^2} = G \frac{\partial^2 \Psi}{\partial x^2} \qquad (2.31)$$

and

$$\frac{\partial^2 \Psi}{\partial x^2} - \frac{\rho}{G}\frac{\partial^2 \Psi}{\partial t^2} = 0. \tag{2.32}$$

By comparing to the wave equation one obtains:

$$\mathrm{v}_t = \sqrt{\frac{G}{\rho}}. \tag{2.33}$$

We have the connection between the modulus of elasticity E and the shear modulus G:

$$G = \frac{E}{2(1+\mu)} \tag{2.34}$$

and it follows:

$$\mathrm{v}_t = \sqrt{\frac{G}{\rho}} = \sqrt{\frac{E}{2(1+\mu)\rho}} < \mathrm{v}_l. \tag{2.35}$$

In the case when the length of the string (solid rod) is infinite (boundless) or with lateral dimensions much larger than the wavelength one gets:

$$\frac{\mathrm{v}_l^{'}}{\mathrm{v}_l} = \sqrt{\frac{2(1-\mu)}{1-2\mu}} > 1. \tag{2.36}$$

In seismology transverse waves are called S (for “secondary”) waves as they arrive later than the P (“primary”) waves from an earthquake. The absence of transverse waves traveling through the earth’s core shows that it is liquid.

Sound will travel slower in denser materials, and faster in "springier" ones. Sound will travel faster in aluminium than uranium, and faster in hydrogen than nitrogen because of the lower density of the first material of each set. At the same time, sound will travel faster in aluminium than hydrogen, as the internal bonds in aluminium are much stronger. Generally solids will have a higher speed of sound than liquids or gases.

2.1.3.2 Propagation in a Fluid

In fluids there are only longitudinal waves. The only non-zero stiffness is to volumetric deformation (a fluid does not sustain shear forces).

Hence the speed of sound in a fluid is given by:

$$\mathrm{v} = \sqrt{\frac{\chi_{adiab}}{\rho}}\,, \tag{2.37}$$

where χ_{adiab} is the adiabatic *bulk modulus*. The bulk modulus *K* of a fluid or solid is the inverse of the compressibility:

$$\chi_{adiab} = -V\frac{\partial p}{\partial V}\,, \tag{2.38}$$

where p is pressure and V is volume. The bulk modulus thus measures the response in pressure due to a change in relative volume, essentially measuring the substance's resistance to uniform compression.

The speed of sound in water is of interest for mapping the ocean floor. In saltwater, sound travels at about 1500 $m{\cdot}s^{-1}$ and in freshwater 1435 $m{\cdot}s^{-1}$. These speeds vary due to pressure, depth, temperature, salinity and other factors.

2.1.3.3 Propagation in Ideal Gases and in Air

In gases there are only longitudinal waves. Newton considered that the propagation of sounds in gases is an isothermal one. He considered the speed of sound before most of the development of thermodynamics. He incorrectly used isothermal calculations instead of adiabatic. The factor of γ does not appear in his result, which was otherwise correct.

The Boyle-Mariotte's law is:

$$pV = const.\,,$$

$$\ln p + \ln V = \ln(const.) = 0\,,$$

$$(\frac{dp}{dV})_{isot} = -\frac{p}{V} \qquad (2.39)$$

and

$$\chi_{isot} = -V(\frac{dp}{dV})_{isot} = p\,, \qquad (2.40.a)$$

the isothermal bulk modulus is equal to the pressure of gas. The speed of sound is given by:

$$\mathrm{v} = \sqrt{\frac{p}{\rho}}\,. \qquad (2.40.b)$$

The experiments do not confirm this equation. Laplace made the connection between theory and experiments. He considered that the propagation of sounds in gases is an adiabatic process. The Poisson law is:

$$pV^{\gamma} = const.$$

$$\gamma = \frac{C_p}{C_V} = \frac{c_p}{c_V} \qquad (2.41)$$

and

$$\ln p + \gamma \ln V = \ln(const.) = 0\,,$$

$$(\frac{dp}{dV})_{adiab} = -\gamma\frac{p}{V}\,,$$

$$\chi_{adiab} = \gamma p = \gamma \chi_{isot} \tag{2.42}$$

and we have the speed of sound in ideal gases and air given by:

$$\mathrm{v} = \sqrt{\frac{\gamma p}{\rho}}. \tag{2.43}$$

Using the ideal gas law Clapeyron-Mendeleev:

$$pV = \frac{m}{\mu} RT \Rightarrow \rho = \frac{m}{V} = \frac{p\mu}{RT}, \tag{2.44}$$

the speed of sound is identical to:

$$\mathrm{v} = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma RT}{\mu}}. \tag{2.45}$$

In the equation above we have R (287.05 $\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$ for air) is the gas constant for air: the universal gas constant R, which units of $\mathrm{J{\cdot}mol^{-1}{\cdot}K^{-1}}$, is divided by the molar mass of air, as is common practice in aerodynamics, μ is the molar mass, γ is the adiabatic index and T is the absolute temperature in kelvins.

For an ideal gas, the speed of sound depends on temperature only, not on the pressure. We can consider that the air is almost an ideal gas. The temperature of the air varies with altitude, this cause the following variations in the speed of sound using the standard atmosphere - actual conditions may vary.

Given normal atmospheric conditions, the temperature, and thus speed of sound, varies with altitude*.

Altitude	**Temperature**	**$m \cdot s^{-1}$**	**$km \cdot h^{-1}$**	**mph**
Sea level	15 °C (59 °F)	340	1225	761
11,000 m–20,000 m (Cruising altitude of commercial jets, and first supersonic flight)	-57 °C (-70 °F)	295	1062	660
29,000 m (Flight of X-43A)	-48 °C (-53 °F)	301	1083	673

In air a range of different methods exist for the measurement of sound, like the single-shot timing methods and Kundt's tube.

2.1.4 Energy of Waves. Wave Intensity.

Propagation of waves means propagation of energy. The particles of the medium oscillate and we have a propagation of the state of motion in the medium.

For the longitudinal waves that propagate in a solid rod with the cross section S, when the length of the solid rod is finite (a bounded solid rod) or with thickness much smaller than the wavelength, if the force that is acting is F and the displacement is $\Psi(x)$, the cross section has the speed:

$$v' = \frac{\partial \Psi}{\partial t}. \tag{2.46}$$

Also, upon the cross section is acting the elastic force $F_{el} = -F$. The mechanical power is:

$$P = \frac{\partial W}{\partial t} = (-F)v' = -F\frac{\partial \Psi}{\partial t}. \tag{2.47}$$

If the elastic wave (sound) is sinusoidal:

$$\Psi = A\sin(\omega t - kx), \tag{2.48}$$

one has:

$$v' = \frac{\partial \Psi}{\partial t} = A\omega\cos(\omega t - kx) \tag{2.49}$$

and from the Hooke's law it follows:

$$\sigma = \frac{F}{S} = E\varepsilon \tag{2.50}$$

and

$$F = SE\varepsilon = SE\frac{\partial \Psi}{\partial x} = -ESAk\cos(\omega t - kx). \tag{2.51}$$

The mechanical power is:

$$P = \frac{\partial W}{\partial t} = ESk\omega A^2 \cos^2(\omega t - kx). \tag{2.52}$$

With the equations:

$$\mathrm{v} = \sqrt{\frac{E}{\rho}}, \qquad \rho \mathrm{v}^2 = E \tag{2.53}$$

and

$$k = \frac{\omega}{\mathrm{v}}, \tag{2.54}$$

the power becomes:

$$P = \frac{\partial W}{\partial t} = \rho \mathrm{v}^2 S \frac{\omega^2}{\mathrm{v}} A^2 \cos^2(\omega t - kx) \tag{2.55}$$

and

$$P = \frac{\partial W}{\partial t} = \rho \mathrm{v} S \omega^2 A^2 \cos^2(\omega t - kx). \tag{2.56}$$

From (2.56) it follows that even the power is time dependent it has always positive values because of $\cos^2(\omega t - kx)$ that has positive values. The power depends on $\omega t - kx$, so it corresponds to a *wave of energy*.

We have the time medium value of the power:

$$\begin{aligned} \overline{P} &= \left(\frac{\partial W}{\partial t}\right)_{med} = \frac{1}{T}\int_0^T \frac{\partial W}{\partial t} dt = \\ &= \frac{1}{T} \mathrm{v} S \rho \omega^2 A^2 \int_0^T \cos^2(\omega t - kx) dt = \frac{1}{2} \mathrm{v} S \rho \omega^2 A^2. \end{aligned} \tag{2.57}$$

For solid rod the product $\mathrm{v}S$ is the volume where the waves have propagated per unit time. We define the quantity:

$$w = \frac{1}{2} \rho \omega^2 A^2, \tag{2.58}$$

that is the *energy density* of waves or wave *energy density* and one has:

$$w = 2\rho\pi^2\nu^2 A^2 \tag{2.59}$$

and it follows:

$$\overline{P} = \mathrm{v}Sw. \tag{2.60}$$

Energy density of waves is the amount of energy of waves stored in a given system or region of space per unit volume.

From (2.60) it results that the energy density of waves is proportional to the square of amplitude, the square of frequency and the linear density.

The sound energy density or sound density describes the sound field at a given point as a sound energy value. The sound energy density describes the time medium value of the sound energy per volume unit; it gives information about the sound energy which is at a defined place of room. The sound energy density is given in J/m^3. For sounds in air we have $w = 2{,}5 \cdot 10^{-5}\ J/m^3$.

The *wave intensity*, I, is defined as the wave power P per unit area:

$$I = \frac{1}{S}\overline{P} = \frac{1}{2}\rho \mathrm{v}\omega^2 A^2. \tag{2.61}$$

Because the maximum of speed is $\mathrm{v}'_{\max} = \omega A$ one gets:

$$I = \frac{1}{2}\rho \mathrm{v}\mathrm{v}'^2_{\max}. \tag{2.62}$$

From (2.62) it results that the wave intensity is proportional to the wave energy density and the wave velocity. For sounds in air we have $I = 2{,}5 \cdot 10^{-5}\ J/m^3 \cdot 340\ m/s = 8{,}5 \cdot 10^{-3}\ W/m^2$.

*Some Applications of the Wave*s*:

The terms of wave power refers to the energy of ocean surface waves and the capture of that energy to do useful work - this includes electricity generation, desalination, and the pumping of water (into reservoirs).

Wave power can be considered a form of renewable energy and it is different than the diurnal flux of tidal power and the steady gyre of ocean currents.

Large waves are more powerful and the wave power is determined by wave height, wave speed, wavelength, and water density.

Wave size is determined by wind speed and fetch and by the depth and topography of the seafloor. For a given wind speed there is a practical limit over which time or distance will not produce larger waves. This limit is called a "fully developed sea."

Wave motion is highest at the surface and increases exponentially with depth. Wave energy is also present as pressure waves in deeper water. For a set of waves the potential energy is proportional to wave height squared times wave period (the time between wave crests). Because longer period waves have relatively longer wavelengths they move faster. The potential energy is equal to the kinetic energy. Wave power is expressed in kilowatts per meter.

Fig. 1.30

Pelamis machine pointing into the waves: it attenuates the waves, gathering more energy than its narrow profile suggests*

The Pelamis Wave Energy Converter represents an emerging technology that will use the motion of ocean waves for producing electricity. The first "wave farm" was planned for 2006 off the coast of Portugal. The wave farms used 3 Pelamis P-750 machines, each of them being capable of producing 750 kilowatts, and each farm producing 2.25 megawatts.

2.2 Wave Interference

Interference is a phenomenon that consists in the superposition of two or more waves resulting in a new wave pattern.

In order to have interference of waves these must be correlated or coherent with each other, either because they have the same source or because they have the same or nearly the same frequency. Two non-monochromatic waves are coherent with each other if they both have the same wavelength (the same frequency) and the same phase differences at each of the constituent wavelengths. Many waves do not obey to these conditions, so it is necessary to make them coherent with each other for having interference.

Important is the principle of superposition of waves: the resultant displacement at a point is equal to the sum of the displacements of different waves at that point.

There are two important cases of interference: *constructive interference* and *destructive interference. Constructive interference*: superposition of two crests belonging to different waves at the same point with the increasing of the resultant wave amplitude. *Destructive interference*: superposition of two crests belonging to different waves at the same point with the decreasing of the resultant wave amplitude.

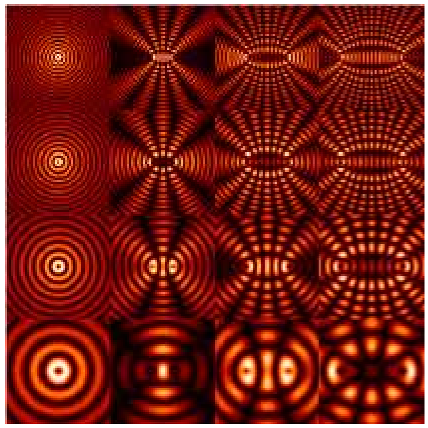


Fig. 1.31

Interference of two circular waves*

Examples of interference: interference of sounds, interference of surface waves on the water, interference of gravitational waves, and interference of light.

Examples of interferences of light: light from any source can be used to obtain interference patterns, Thomas Young's double-slit experiment, and Newton's rings that can be produced with sunlight.

About the white light, we notice that it is not so suitable for producing clear interference patterns, because is a mix of colours, that each has different spacing of the interference fringes. An example of light close to monochromatic is the sodium light. This is more suitable for obtaining interference patterns. The laser light exhibits the same property, and is almost perfectly monochromatic.

In the case of the interference of two waves, the resulting waveform depends on the frequency (or wavelength) amplitude and relative phase of the two waves. If we have two waves of the same amplitude A and wavelength the resultant waveform will have amplitude between 0 and $2A$. The first case corresponds to two waves that are in phase, and the second case describes two waves that are out of phase. In the figure below are presented two waves in phase and two waves 180° out of phase.

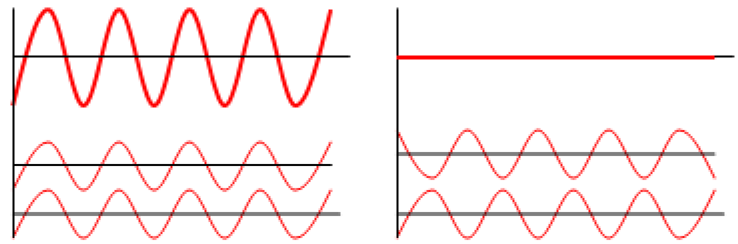


Fig. 1.32

Two waves in phase and two waves 180° out of phase*

If these two waves are in phase, and have the amplitudes A_1 and A_2, then their troughs and peaks line up and the resultant wave has the amplitude $A = A_1 + A_2$. This is an example of *constructive interference*.

If the two waves are 180° out of phase, then one wave's crests will coincide with another wave's troughs, and they will tend to cancel out each other. The resultant amplitude is $A = |A_1 - A_2|$. For equal amplitudes $A_1 = A_2$ the resultant amplitude will be zero. This is an example of *destructive interference*.

*Thomas Young's *double-slit experiment* (Fig. 1.33) is based on the phenomenon of interference, the case of two beams of light which are coherent with each and produce an interference pattern (the beams of light have the same wavelength range and both come from the same source). Two or more sources can produce interference when there is a fixed phase relation between them, but in the case of this experiment the interference generated is the same as with a single source.

The light diffracted through two slits produces fringes on a screen. These fringes present light and dark regions that correspond to constructive and destructive interference. The experiment can be realized with a beam of electrons or atoms, and in this case similar interference patterns can be obtained; this is an evidence of the "wave-particle duality" predicted by quantum physics. Also, a double-slit experiment can be performed with water waves in a ripple tank; for the explanation of the interference there is no need of quantum mechanics. The phenomenon is quantum mechanical only when quantum particles - such as atoms, electrons, or photons - manifest as waves. The condition for obtaining an interference pattern in a double-slit experiment concerns the difference in path length between two paths that light can take to reach a zone of constructive interference on the viewing screen. This difference has to be equal to the wavelength of the light, or a multiple of this wavelength.

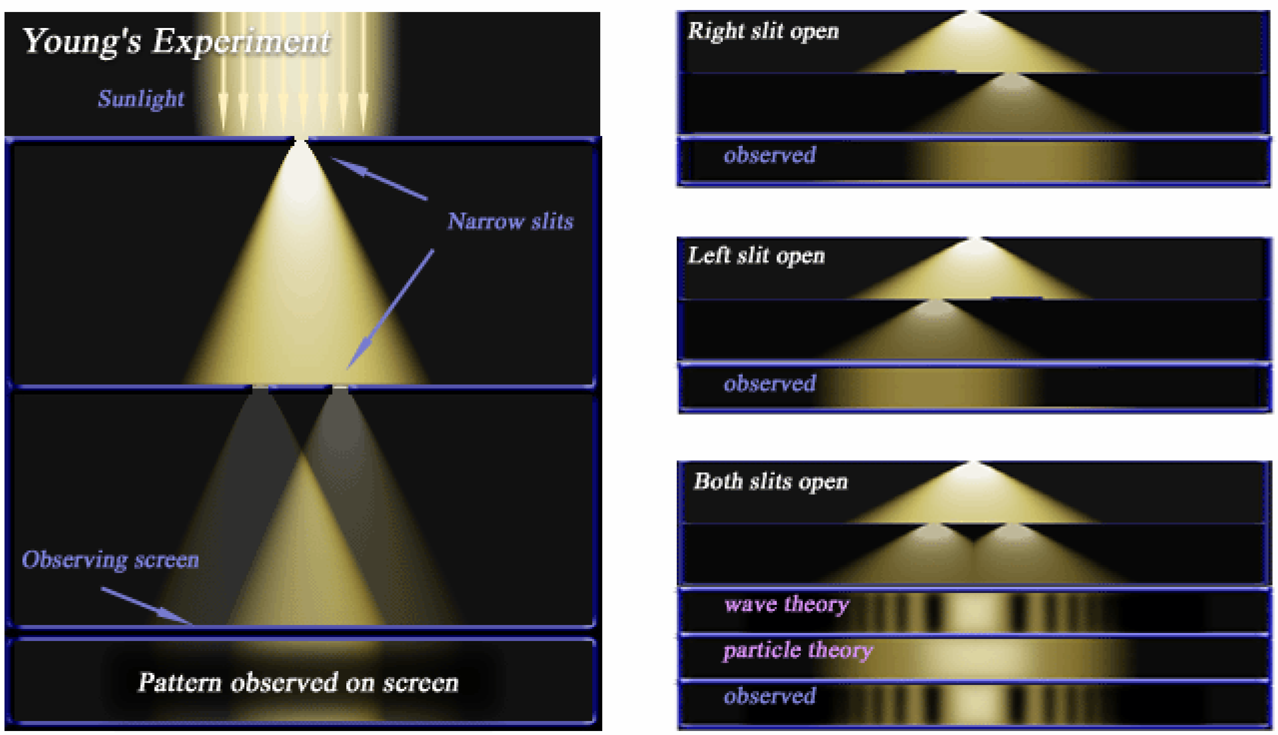


Fig. 1.33*

In Young's original experiment, Sunlight passes first through a single slit, and then through two thin vertical slits in otherwise solid barriers, and the interference pattern is viewed on a rear screen.

When either slit is covered, a single peak can be observed on the screen, and is caused by the light passing through the other slit. For both slits open simultaneously, a pattern of light and dark fringes is observed.

This pattern of fringes contains constructive interference and destructive interference. The brighter spots are connected to constructive interference, where two peaks in the light wave coincide as they reach the screen. The darker spots are connected to destructive interference that occurs where a peak and a trough occur together. In the figure below we have intensity represented versus position.

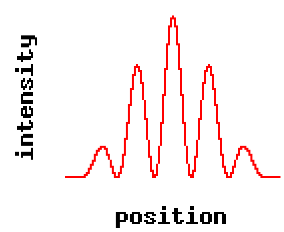


Fig. 1.34

Interference*

One has:

$$\frac{\lambda}{s} = \frac{x}{D} \tag{2.63}$$

where:

λ is the wavelength of the light,

s is the separation of the slits,

x is the distance between the bands of light (fringe distance),

D is the distance from the slits to the screen.

The interference fringes observed in Young's double slit experiment have shapes that are straight lines.

In the original Young's experiment instead of slits are used two pinholes, and hyperbolic fringes are observed.

In the case when the two sources are placed on a line perpendicular to the screen, the shape of the interference fringes is circular as the individual paths travelled by light from the two sources are always equal for a given fringe. This can be obtained by placing a mirror parallel to a screen at a distance and using a source of light that is placed just above the mirror.

2.2.1 Theoretical Demonstration of Interference

We consider waves which are correlated or coherent with each other, either because they come from the same source or because they have the same or nearly the same frequency. The condition of coherence for two non-monochromatic waves is respected only if they both have exactly the same range of wavelengths (the same frequency) and the same phase differences at each of the constituent wavelengths.

We have two sinusoidal waves as in the figure below described by:

$$\begin{aligned}\Psi_1 &= \frac{A_1}{r_1}\sin 2\pi\left(\frac{t}{T}-\frac{r_1}{\lambda}\right),\\ \Psi_2 &= \frac{A_2}{r_2}\sin 2\pi\left(\frac{t}{T}-\frac{r_2}{\lambda}\right).\end{aligned} \tag{2.64}$$

We consider that the initial phases vanish. The point M is far away from the sources S_1 and S_2 (Fig. 1.35) and the waves have the same of propagation.

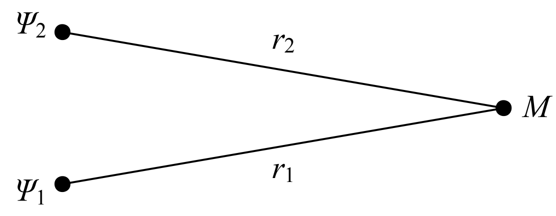


Fig. 1.35

We denote $a_1 = \frac{A_1}{r_1}$ and $a_2 = \frac{A_2}{r_2}$ and we make the approximation $a_1 \cong a_2 = a$ and we obtain for the resultant wave:

$$\Psi = \Psi_1 + \Psi_2 = 2a \cos \pi \frac{r_2 - r_1}{\lambda} \sin 2\pi\left(\frac{t}{T} - \frac{r_1 + r_2}{2\lambda}\right). \tag{2.65}$$

The equation

$$r_1 + r_2 = const. \tag{2.66}$$

represents the locus of equal phase points (equi-phase surface) which is the equation of a family of ellipsoids revolving around the line S_1S_2 having the two focuses in the points where the sources S_1 and S_2 are placed.

The surface given by the equation:

$$r_2 - r_1 = const. \tag{2.67}$$

is the place of the space points which have the resultant amplitude $\left|2a \cos \pi \frac{r_2 - r_1}{\lambda}\right|$ constant.

This relation is the equation of a family of revolving hyperboloids around the axis S_1S_2 having the two focuses in the points where the sources S_1 and S_2 are placed.

The resulting amplitude corresponds to a maximum when the displacement is a multiple of wave lengths or an even multiple of half-wave lengths:

$$r_2 - r_1 = 2m\frac{\lambda}{2} = m\lambda, \tag{2.68}$$

with $(m = 0,1,2,...)$.

The resulting amplitude corresponds to a minimum when the displacement is a multiple of wave lengths or an odd multiple of half-wave lengths:

$$r_2 - r_1 = (2m+1)\frac{\lambda}{2}, \tag{2.69}$$

with $(m = 0,1,2,...)$.

The interference phenomenon (waves which are correlated or coherent with each other) is easy to observe in the case of the surface waves made by water when two small stones are thrown simultaneously at a certain distance one from other. Other examples of interference are the cases of the water waves emitted by the tips of two elastic blades that touch the water surface at the same time, the sound waves emitted by two diffusers controlled by the same sound oscillator or light emitted by the currents of the margins of two slits that are lighted by a linear point like source.

2.2.2 Doppler Effect

The *Doppler effect*, discovered by Christian Andreas Doppler, and is observed when the source of waves is moving with respect to an observer. The

Doppler effect consists in an apparent change in frequency or wavelength of a wave that is perceived by an observer moving relative to the source of the waves.

In the case of sound waves that propagate in a wave medium the velocity of the observer and the source are considered relative to the medium in which the waves propagate. The total Doppler effect is the result of either motion of the source or motion of the observer. The study of these effects can be realized separately. For light or gravity (waves which do not require a medium for propagation) in special relativity only the relative difference in velocity between the observer and the source is needed for the study of the effect.

The frequency of the sounds emitted by the source does not actually change. For pointing out the Doppler effect we consider an analogy. A ball is thrown every second in an observer's direction. The velocity of the balls is constant. If the thrower is stationary, the observer will receive one ball every second. In the case when the thrower is moving towards the observer, he will receive balls more frequently because the balls will be less spaced out. This is also available if the thrower is moving away from the observer. The *wavelength* will be affected, and a consequence is that the perceived frequency is also affected.

Applications of Doppler Effect[*]:

1) The Doppler effect for electromagnetic waves such as light has applications in astronomy, and is connected to either a redshift or blueshift. In physics and astronomy, Redshift represents an observed increase in the wavelength (decrease in the frequency) of electromagnetic radiation that is received by a detector compared to that emitted by the source. In the case of visible light the longest wavelength is for res, so colors experiencing redshift shift towards the red part of the electromagnetic spectrum. The phenomenon also occurs at non-optical

wavelengths (longer-wavelength radiation "redshifts" away from red). In the case of shorter wavelengths the corresponding shift is called blueshift. It has been used for the evaluation of the speed (radial velocity) at which stars and galaxies are approaching to, or receding from us. This is used to detect a single star is, in fact, a close binary (a binary star system consists of two stars both orbiting around their center of mass) and even to measure the speed of rotation of stars and galaxies.

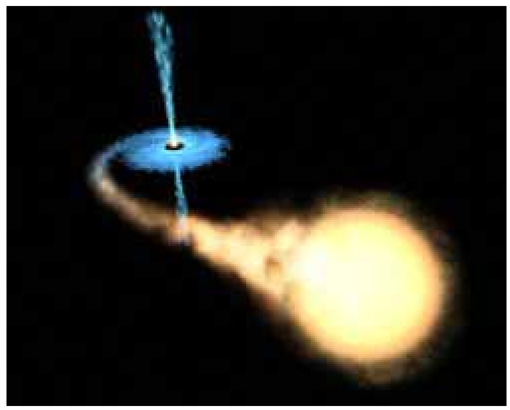


Fig. 1.36

A binary star system consisting of a black hole, with an accretion disc around it, and a main sequence star*

An accretion disk is a structure that is formed by material falling into a gravitational source. Accretion disks are phenomenon in astrophysics; active galactic nuclei, protoplanetary discs, and gamma ray bursts.

The Doppler effect for light is used in astronomy because the spectra of stars are not continuous. This implies the existence of absorption lines at well defined frequencies that are correlated with the energies required to excite electrons in various elements from one level to another. A characteristic of the Doppler effect is that the absorption lines are not always at the frequencies obtained from the spectrum of a stationary light source. Blue light has a lower wavelength than red light, and the spectral lines from an approaching astronomical light source show a blueshift and those of receding sources show a redshift.

2) Another application of the Doppler effect in astronomy is *temperature measurement* for a gas which is emitting a spectral line The thermal motion of the gas, determines that each emitter can be slightly red or blue shifted, and the net effect is a broadening of the line. The line shape is called a Doppler profile and its width is proportional to the square root of the temperature of the gas. This allows the use of the Doppler-broadened line for measuring the temperature of the emitting gas.

3) The Doppler effect can be used in some forms of radar to measure the velocity of detected objects. A radar beam is fired at a moving target (a car) as it recedes from the radar source. For being reflected by the car and re-detected near the source each wave has to travel further to reach the car. There will be an increasing of the gap between each wave, and this will imply an increasing of the wavelength. The radar beam is fired at a moving car that approaches, and in this case the successive wave travels a decreased distance, and this determines the decreasing of the wavelength. The car's velocity can be determined with the Doppler effect.

4) The laser Doppler velocimeter (LDV), and Acoustic Doppler Velocimeter (ADV) are used to measure velocities in a fluid flow. The LDV and ADV emit a light or acoustic beam, and allow the evaluation of the Doppler shift in wavelengths of reflections from particles moving with the flow. This technique is used for flow measurements, at high precision and high frequency.

2.3 Sounds

2.3.1 Sound Intensity

The sound represents the propagation of the vibrational mechanical energy through matter in a form of a wave. Hearing is possible between frequencies about 20 Hz and 20000 Hz, with the upper limit generally decreasing with age, in the case of humans. The propagation of mechanical vibrations takes place through gases, liquids, solids, and plasmas. Through solids the propagation is made as both like longitudinal and transverse waves (shear waves) and through gases, liquids and plasma as longitudinal waves (compression waves).

The *sound intensity*, I, (acoustic intensity) is defined as the sound power P per unit area S. The SI units are W/m^2.

The sound intensity (of a plane progressive wave) is given by:

$$I = \frac{1}{S}\overline{P}. \tag{2.70}$$

For a spherical sound source, the intensity as a function of distance r is:

$$I = \frac{\overline{P}}{4\pi r^2}. \tag{2.71}$$

From (2.62) and (2.70) it results that the sound intensity is proportional to the wave energy density and the wave velocity. For sounds in air we have $I = 2{,}5 \cdot 10^{-5}\ J/m^3 \cdot 340\ m/s = 8{,}5 \cdot 10^{-3}\ W/m^2$.

The amplitude of sound intensity decreases in the free field (direct field) with $\frac{1}{r^2}$ of the distance of a point source. *Sound intensity level* or *acoustic intensity level* is a logarithmic measure of the sound intensity in comparison to the reference level of 0 dB (decibels).

$$L = 10\,\lg\frac{I}{I_0}\,(dB), \tag{2.72}$$

where $I_0 = 10^{-12}\,W/m^2$ (at $\nu = 1\,kHz$) is the reference intensity. Also, we can write for the sound intensity leve:l

$$L = \lg\frac{I}{I_0}\,(B), \tag{2.73}$$

with B from Bell or

$$L = \ln\frac{I}{I_0}\,(Np), \tag{2.74}$$

with Np from Neper.

Sound intensity is different from sound pressure. Hearing is sensitive to sound pressure which is connected to sound intensity.

Sound pressure is the local pressure deviation from the average pressure determined by a sound wave. Measurements of sound pressure can be made using a microphone in air and a hydrophone in water. The SI unit for sound pressure is the pascal (symbol: Pa). A microphone is an acoustic to electric transducer that converts sound into an electrical signal. Microphones are used in many applications: telephones, tape recorders, hearing aids, motion picture production, live and recorded audio engineering, in radio and television broadcasting and in computers for recording voice, VoIP (Voice over Internet Protocol) and numerous other computer applications. A hydrophone is a sound-to-electricity transducer for use in water or other liquids, and represents the analogous to a microphone for air. Hydrophones are an important part of the SONAR (SOund Navigation And Ranging) that is a technique based on the use of sound propagation under water to navigate or to detect other vessels system. Also, geologists and geophysicists use the hydrophones for detecting seismic energy. They are combined to form streamers that are towed by seismic vessels or deployed in a borehole (is a deep and narrow shaft in the ground used for abstraction of fluid or gas reserves below the earth's surface. In the case when the fluid reserve is under pressure (oil or gas) then little extra machinery is required. For water is used a special submersible pump to pump water up the rising main.

Sound pressure level (SPL) is given by:

$$L = 10\,\lg \frac{I}{I_0}(dB) = 20\,\lg \frac{p_s}{p_{s0}}(dB), \tag{2.75}$$

where $p_{s0} \cong 2 \cdot 10^{-5}\,N/m^2$ is the reference sound pressure which corresponds to $I_0 = 10^{-12}\,W/m^2$. This expression is used for sound pressure when dealing with hearing, as the perceived loudness of a sound is connected roughly logarithmically to its sound pressure.

The *Weber–Fechner law* yields the relationship between the physical magnitudes of stimuli and the perceived intensity of the stimuli. The Weber–Fechner law is given by:

$$S - S_0 = k\,\lg \frac{I}{I_0}, \tag{2.76}$$

where k represents a constant factor that can be determined experimentally, S is the stimulus at the instant and S_0 is the threshold of stimulus below which it is not perceived at all.

When making measurements in air (and other gases), SPL is almost always expressed in decibels compared to a reference sound pressure of 20 μPa (micropascals), which is usually considered the *threshold of human hearing* (roughly the sound of a mosquito flying 3 metres away). Most measurements of audio equipment can be made relative to this level. In other media (underwater) a reference level of 1 µPa is used. About the threshold of human hearing, it is the sound pressure level SPL of 20 µPa (micropascals) = 2×10^{-5} pascal (Pa). This low threshold of amplitude (strength or sound pressure level) has a dependence on frequency. See the frequency curve in the figure bellow

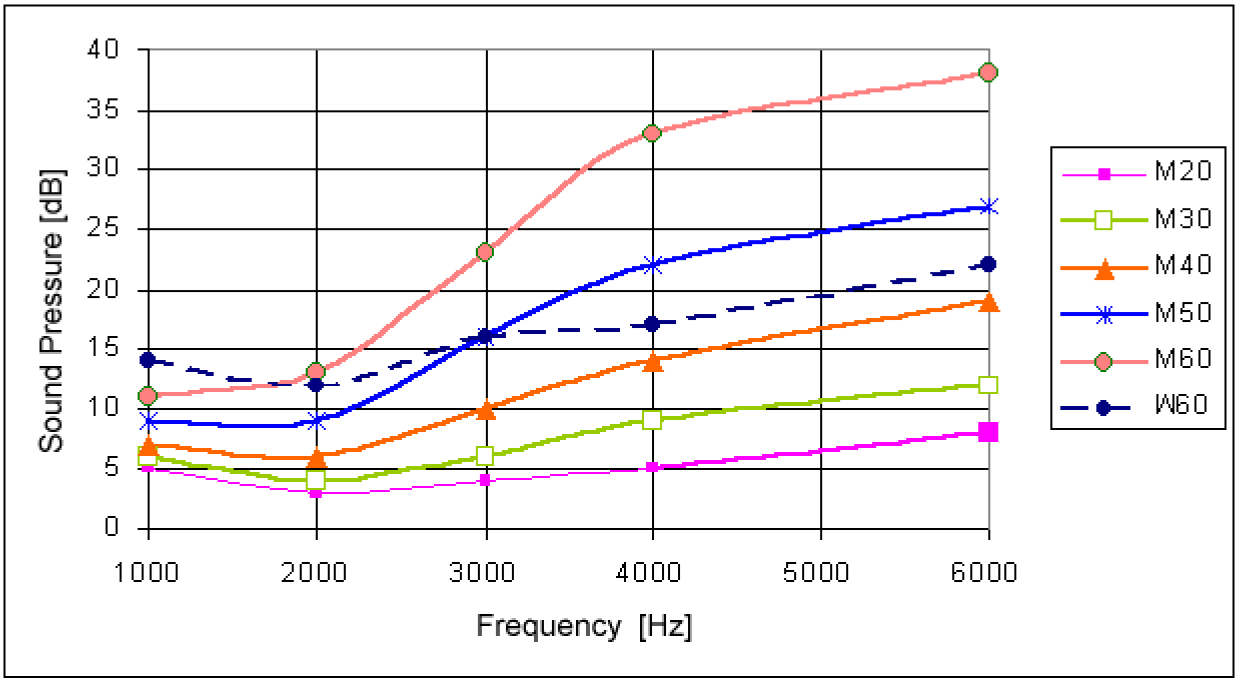


Fig. 1.37

Thresholds of hearing for male (M) and female (W) subjects between the ages of 20 and 60*

The *absolute threshold of hearing* (ATH) represents the minimum amplitude of a pure tone that can be heard by the average ear with normal hearing in a noiseless environment.

The *threshold of pain* is the SPL beyond which sound becomes unbearable for a human listener. This threshold has a dependence on frequency. We give some values for the threshold of pain.

Threshold of pain	
SPL	**sound pressure**
120 dBSPL	**20 Pa**
130 dBSPL	63 Pa
134 dBSPL	**100 Pa**
137.5 dBSPL	150 Pa
140 dBSPL	200 Pa

The threshold of hearing is frequency dependent, and has a minimum (indicating the ear's maximum sensitivity) at frequencies between 1 kHz and 5 kHz. The lowest curve amongst the set of equal-loudness contours represents the absolute threshold of hearing, and the highest curve represents the threshold of pain.

Together with masking curves the ATH is used in psychoacoustic audio compression for evaluating which spectral components are inaudible and may thus be ignored in the coding process. The part of an audio spectrum which has an amplitude (level or strength) below the ATH may be removed from an audio signal without changing the signal. Because of the age (the human ear becomes more insensitive to sound) the ATH curve rises and presents the greatest changes occurring at frequencies higher than 2 kHz.

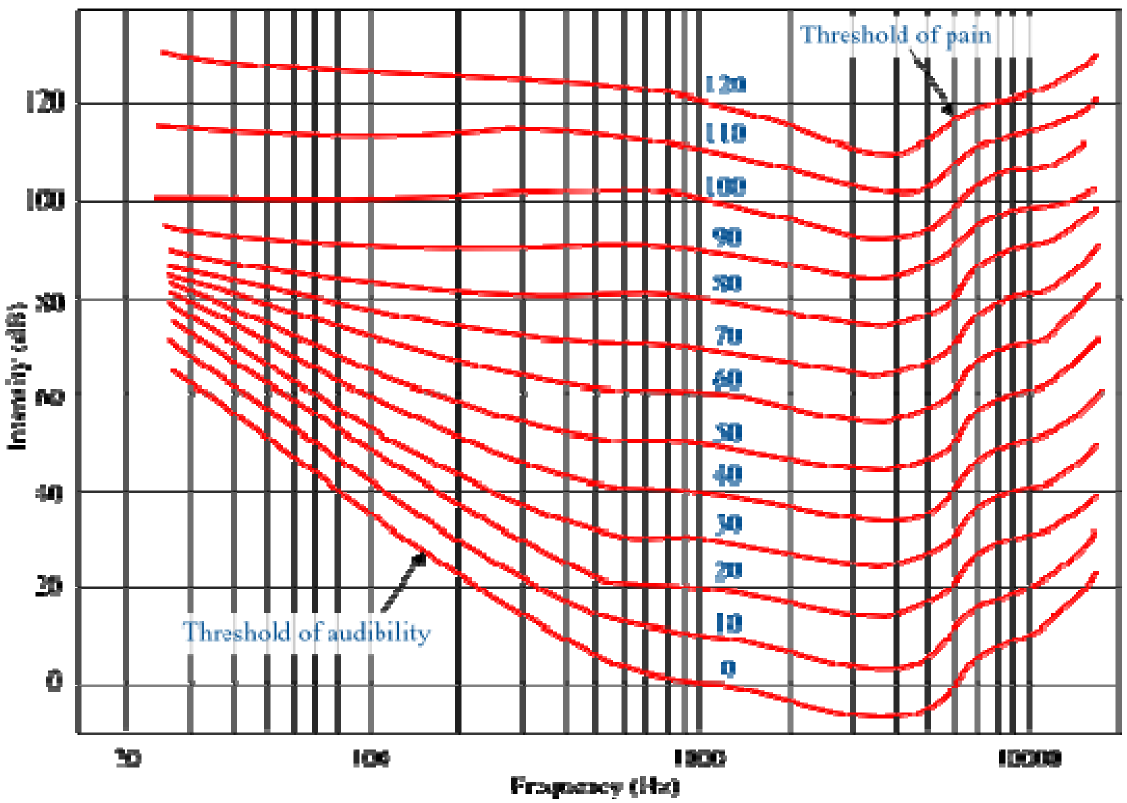


Fig. 1.38

The Fletcher-Munson equal-loudness contours. The lowest of the curves is the ATH*

Some examples of sound pressure levels are given in the table

situation	sound pressure level dBSPL
threshold of pain	130
hearing damage during short term effect	from 120
jet, 100 m distant	110–140
jack hammer, 1 m distant / discotheque	approx. 100
hearing damage during long-term effect	approx. 90
major road, 10 m distant	80–90
passenger car, 10 m distant	60–80
TVset at home level, 1 m distant	approx. 60
normal talking, 1 m distant	40–60
very calm room	20–30
leaves noise, calm breathing	10

2.3.2 Ultrasounds

Ultrasound is sound with a frequency greater than the upper limit of human hearing, with a limit at approximately 20 kilohertz (20,000 hertz). Ultrasounds also represent propagation of acoustic energy in the form of waves. Some animals are able to hear ultrasound (dogs, dolphins, bats, and mice) because they have an upper limit that is greater than that of the human ear. Children can hear some high-pitched sounds that older adults are not able to hear because in humans the upper limit pitch of hearing gets lower with the increasing of age. The middle ear that acts as a low-pass filter is responsible of the frequency limit. In the case when ultrasound is coupled directly into the skull bone and reaches the cochlea and don't pass through the middle-ear, it is possible to hear sounds with frequencies up to about 200 kHz. This effect (ultrasonic hearing) was first discovered by divers exposed to a high-frequency (50 kHz) sonar signal.

Applications of ultrasounds:

1) Ultrasounds are used in medical ultrasonography (sonography) that is an ultrasound-based diagnostic imaging technique. Sonography is used to visualize muscles and internal organs, and to study their size, structure and any pathological lesions. During pregnancy obstetric sonography is used for monitorizing the baby. Ultrasound also has therapeutic applications:

2) Ultrasounds are used for treating benign and malignant tumors and other disorders, via a process called Focused Ultrasound Surgery **(**FUS**)** or HIFU, High Intensity Focused Ultrasound. Lower frequencies than medical diagnostic ultrasound (from 250 kHz to 2000 kHz) are used coupled with significantly higher time-averaged intensities. The treatment is often guided by MRI, as in Magnetic Resonance guided Focused Ultrasound.

3) For cleaning the teeth in dental hygiene or generate local heating in biological tissue (physical therapy and cancer treatment) can be used more powerful ultrasound sources.

4) Focused ultrasound sources are suitable for cataract treatment by phacoemulsification.

5) Low-intensity ultrasounds are used for producing stimulation of bone-growth and to disrupt the blood-brain barrier for drug delivery.

6) Ultrasound-assisted lipectomy UAL or liposuction is based on the use of ultrasound.

7) Ultrasounds have also industrial applications they are used in industry for locating flaws in materials. Most common are ultrasounds with frequencies of 2 to 10 MHz but for special purposes other frequencies can be used. It is possible to inspect most of the metals, plastics and aerospace composites.

8) Ultrasonic cleaners are based on ultrasounds of frequencies from 20-40 kHz and used for jewellery, lenses and other optical parts, watches, dental instruments, surgical instruments and industrial parts.

9) Some sorts of ultrasound allow the disintegration of biological cells including bacteria. This is used in biological science and in killing bacteria in sewage (subset of wastewater that is contaminated with faeces or urine, but is often used to mean any waste water). Sewage includes domestic, municipal, or industrial liquid waste products disposed of, usually via a pipe or sewer or similar structure.

11) Ultrasounds have applications in sonar systems to determine the depth of the water in a place, to find fishs, to locate submarines, and to detect the presence of SCUBA divers.

10) Ultrasounds are used in ultrasonic intrusion detection system.

Chapter 3.

Fluids

3.1.1 Fluids Properties.

Fluid statics (*hydrostatics)* is the science of fluids at rest. Fluids exhibit the properties of not resisting deformation and the ability to flow. These properties are connected to their inability to support a shear stress in static equilibrium. In a fluid stress is a function of rate of strain. Pascal's law is a consequence of this behavior and points out the role of pressure in characterizing a fluid's state. Fluid hydrodynamics studies the fluids in motion.

Fluids can be characterized as:

- *newtonian fluid* is the fluid that exhibits properties of flowing like water, its shear stress is linearly proportional to the velocity gradient in the direction perpendicular to the plane of shear. The constant of proportionality is the viscosity.
- non-newtonian fluid is a fluid that presents a change in the viscosity with the applied strain rate. Because of this a non-Newtonian fluids does not have a well-defined viscosity.

Buoyancy is an upward force that acts on an object immersed in a fluid (a liquid or a gas), allowing it to float or at least to appear lighter. Buoyancy is important for many vehicles such as boats, ships, balloons, and airships.

Buoyancy provides an upward force on the body. According to Newton's first law of motion, if the upward forces (including the buoyancy) balance the downward forces (including the weight) the object will either remain at rest or remain in motion at a constant rate. Otherwise, it will accelerate upwards or downwards.

In the case when an object's compressibility has a value less than that of the surrounding fluid, the object is in stable equilibrium and will remain at rest. If its compressibility is greater, its equilibrium is unstable, and the object will rise and expand on the slightest upward perturbation or fall and compress on the slightest downward perturbation. The condition needed for an object to float is to be able to displace enough water equal to its weight.

The law of buoyancy given by Archimede, also called Archimede's law is: the buoyant force is equal to the weight of the displaced fluid. The weight of the displaced fluid is directly proportional to the volume of the displaced fluid. In the case of more objects with equal masses, the greater buoyancy corresponds to the object with greater volume.

An homogeneous fluid is described by its density that is defined as ratio between mass m and volume V, $\rho = \frac{m}{V}$ (kg/m^3). For a non-homogeneous fluid the equation above describes only the average density. It is necessary to define the density of a given point, in an infinitesimal volume dV with the mass dm. We get $\rho = \frac{dm}{dV}$. Generally, the density is given by $\rho = \rho(x, y, z, y) = \rho(\vec{r}, t)$. The function $\rho(\vec{r}, t)$ describes a density field which is a scalar field. The highest density known is reached in neutron star matter. We give some values of densities of various substances:

Substance	**Density in kg·m^{-3}**	Tin	7310
Iridium	22650	Titanium	4507
Osmium	22610	Diamond	3500
Platinum	21450	Basalt	3000
Gold	19300		
Tungsten	19250	Granite	2700
Uranium	19050		
Mercury	13580	Aluminium	2700
		Graphite	2200
Palladium	12023		
Lead	11340	Magnesium	1740
Silver	10490	***PVC***	1300
Copper	8960		
Iron	7870	***Seawater***	1025
Steel	7850	***Water***	1000
Ice	917	***Ice***	917
Polyethylene	910	***Ethyl alcohol***	790
Gasoline	730	Liquid ***Hydrogen***	68
		Aerogel	3

The forces that act upon a fluid are internal forces and external forces. External forces can be surface forces (from an external body) or volume forces (act upon the whole volume of the fluid: the action of the gravitational field upon a fluid, the action of the electromagnetic field upon a fluid). Internal forces are forces that appear because of the interaction between the infinitesimal volumes of the fluid. Also, they can be surface forces and volume forces.

A liquid is one of the four main phases of matter. It is a fluid with a shape determined by the container it fills. Under conditions of constant temperature and pressure its volume is fixed. The pressure exerted by the liquids on the sides of a container is transmitted undiminished in all directions and increases with depth.

The fluid mechanics has two parts, fluid dynamics and fluid statics depending on whether the fluid is in motion or not.

3.1.2 Fluids Pressure.

Pressure p is the force per unit area that acts on a surface in a direction perpendicular to that surface. It is given by:

$$p = \frac{F}{S}, \tag{3.1}$$

where F is the normal force and S is the area of the surface. This equation is available when the force is the same in every point of the fluid. When the force has different values in different points of the fluid the pressure is given by:

$$p = \frac{dF}{dS}, \tag{3.2}$$

where dF is also the normal force and the surface can be represented by a vector $d\vec{S} = \vec{n}\,dS$, where $\vec{n}$ is the unit (surface normal) vector of the orthogonal direction to dS.

If the pressure has different values in different points of the fluid we have $p = p(x, y, z, t) = p(\vec{r}, t)$. The function $p(\vec{r}, t)$ describes a pressure field which is a scalar field.

If the fluid is moving we need a velocity fields to describe it. The velocity of the fluid is given by $\vec{v} = \vec{v}(x, y, z, y) = \vec{v}(\vec{r}, t)$. The function $\vec{v}(\vec{r}, t)$ describes a velocity field which is a vector field.

The unit for pressure in SI is pascal Pa , $Pa = N/m^2$. Other unit for pressure is the standard atmosphere atm which is given by, $1atm = 1.01325 \cdot 10^5\ Pa = 1.01325 \cdot 10^5\ N/m^2$. In the table bellow we present the pressure units*.

Pressure Units

	Pascal (Pa)	**Bar (bar)**	**Technical atmosphere (at)**	**Atmosphere (atm)**	**Torr (mmHg)**	**Pound per square inch (psi)**
1 Pa	≡ 1 N/m²	10^{-5}	10.197×10^{-6}	9.8692×10^{-6}	7.5006×10^{-3}	145.04×10^{-6}
1 bar	100 000	≡ 10^6 dyn/cm²	1.0197	0.98692	750.06	14.504
1 at	98 066.5	0.980665	≡ 1 kgf/cm²	0.96784	735.56	14.223
1 atm	101 325	1.01325	1.0332	≡ 101 325 Pa	760	14.696

1 Torr	133.322	1.3332×10^{-3}	1.3595×10^{-3}	1.3158×10^{-3}	≡ 1 mmHg	19.337×10^{-3}
1 psi	6 894.76	68.948×10^{-3}	70.307×10^{-3}	68.046×10^{-3}	51.715	≡ 1 lbf/in²

Pressure is measured by its ability to displace a column of liquid in a manometer, and often expressed as a depth of a particular fluid. The most used choices are mercury (Hg) and water. Water has the properties of no toxicity and readily availability, and mercury's density gives the possibility to use a shorter column (a smaller manometer) to measure a given pressure.

3.1.3 Fluid Statics. Hydrostatic Pressure.

Fluid pressure is the pressure on an object submerged in a fluid, such as water. The concept of fluid pressure is associated to the discoveries of Blaise Pascal and Daniel Bernoulli.

The fundamental law of the fluid statics (in the case when upon the fluid are acting only gravitational forces) is given by:

$$\vec{g} = \frac{1}{\rho}\nabla p, \tag{3.3}$$

If upon the fluid also act other external volume forces and the resultant force is $\vec{f}$, the fundamental law of the fluid statics becomes:

$$\vec{f} = \frac{1}{\rho}\nabla p. \tag{3.4}$$

This is a differential equation with first order partial derivatives and if we integrate it we obtain the pressure field $p = p(x, y, z)$. The solution is completely known if there are given the boundary condition for the fluid and the laws of dependence on position for $\vec{f}$ and ρ.

The hydrostatic pressure is an important application of the fundamental law of the fluid statics for an incompressible fluid. For $\vec{g} = \vec{g}(0{,}0{,}-g)$ from the fundamental law of the fluid statics (3.3) it follows:

$$\frac{\partial p}{\partial x} = 0, \frac{\partial p}{\partial y} = 0, \frac{\partial p}{\partial z} = -\rho g. \tag{3.5}$$

The pressure does not depend on the coordinates x and y and is a function on z coordinate and one has:

$$dp = -\rho g dz. \tag{3.6}$$

In the case where the fluid is at rest, the force acting on the object is the sheer weight of the fluid above, up to the water's surface-such as from a water tower. The resulting *hydrostatic pressure* (*static pressure*) is isotropic: the pressure acts in all directions equally, according to Pascal's law:

$$p = \rho g h, \tag{3.7}$$

where ρ is the density of the fluid, g is the acceleration due to gravity (practical value 9.8 m/s^2) and h is the height of the fluid column ($p_2 = p, p_1 = 0, h = z_2 - z_1$). Pascal's law on gives the fluid pressure at mechanical equilibrium. Also, according to the Pascal's law we have the difference of pressure between two different heights h_1 and h_2 given by:

$$\Delta p = p_2 - p_1 = \rho g (h_1 - h_2), \tag{3.8}$$

where h_1 and h_2 are elevations. The elevation of a geographic location is given by its height above a fixed reference point that is in many cases the mean sea level. If we have a fluid in a receptacle and h_1 and h_2 are the depths of two points A and B the equation (3.8) gives the difference of hydrostatic pressure between these two points. The hydrostatic pressure does not depend on the shape of receptacle, only depends on the depth. Also, from (3.8) it follows:

$$p_2 = p_1 + \rho g (h_1 - h_2). \tag{3.9}$$

If the external volume force field which acts upon the fluid is a conservative field (gravitational field) we have $\vec{F} = -\nabla U$ and (3.4) becomes:

$$\vec{f} = \frac{\vec{F}}{m} = -\frac{1}{m}\nabla U = -\nabla(\frac{U}{m}) = -\nabla V, \tag{3.10}$$

where $V = U/m$ is the potential of the external field. From (3.4) and (3.10) we obtain:

$$\frac{1}{\rho}\nabla p + \nabla V = \nabla(\frac{p}{\rho} + V) = 0 \tag{3.11}$$

and one gets:

$$\frac{p}{\rho} + V = const. \tag{3.12}$$

From (3.12) it results that the surfaces of the same pressure are also surfaces of the same potential (equipotential surfaces). In the case from above the equipotential surfaces are horizontal planes.

Applications of Hydrostatic Pressure[*]

- The pressure under water increases with depth, and this is well known to scuba divers. At a depth of 10 m under water, pressure has a double value than the atmospheric pressure at sea level, and increases by 100 kPa for every extra 10 m of depth.

- Any change in pressure applied at any given point on a confined and incompressible fluid is transmitted undiminished throughout the fluid.
- Atmospheric pressure has a decreasing with height, and this was first verified on the Puy-de-Dôme and the Saint-Jacques Tower in Paris by Blaise Pascal. As the atmosphere becomes lighter with height, the atmospheric pressure has an exponentially dependence on height. This is expressed with the barometric formula.
- Artesian wells, water towers, dams.
- Pascal's barrel experiment: in this experiment the main part was a long and narrow vertical pipe connected to the content of a large barrel. Putting water into the pipe (even in small quantity), the height of the fluid within the pipe will sharply increase, and the break of the barrel can be induced.
- Pascal's principle underlies the Hydraulic press.

3.1.4 Fluid Dynamics.

Fluid dynamics studies fluids (liquids and gases) that are in motion.

Also, fluid dynamics yields a mathematical structure that embraces empirical and semi-empirical laws, derived from flow measurement, used to solve practical problems. The solution of a fluid dynamics problem typically involves carry-on calculations for various properties of the fluid, such as velocity, pressure, density, and temperature, as functions of space and time.

The fluid flow can be *compressible flow* and *incompressible flow*. In the case of the *compressible flow* the density of the flow cannot be assumed to be constant. The *incompressible flow* describes a fluid flow in which changes in the fluid density have little effect on the variables of interest, such as the lift (consists of the sum of all the fluid dynamic forces on a body perpendicular to the direction of the external flow approaching that body) on a wing.

3.1.4.1 The Continuity Equation

The *volumetric flow rate* or volume flow rate is given by:

$$Q_V = \frac{dV}{dt} \, (m^3 / s) . \tag{3.13}$$

The mass flow rate is given by:

$$Q_m = \frac{dm}{dt} \, (kg / s) . \tag{3.14}$$

The connection between (3.13) and (3.14) is:

$$Q_m = \rho \frac{dV}{dt} = \rho Q_V . \tag{3.15}$$

One of the fundamental equations of fluids flow is the *continuity equation* which expresses the law of conservation of matter. We consider a fluid in motion, with volume V and the closed surface S around it. The mass of fluid in this volume is:

$$m = \int_{(V)} \rho dV , \tag{3.16}$$

with $\rho = \rho(\vec{r}, t)$ the density of the fluid. From (3.16) one has:

$$\frac{\partial m}{\partial t} = -\frac{\partial}{\partial t} \int_{(V)} \rho dV = -\int_{(V)} \frac{\partial \rho}{\partial t} \, dV . \tag{3.17}$$

On the other hand we have the volume $m = \rho \mathrm{v} dS$ or, generally, $m = \rho \vec{\mathrm{v}} \cdot d\vec{S}$. The variation of the mass of the fluid in time unit from the whole closed surface S is given by:

$$\frac{\partial m}{\partial t} = \oint_{(S)} \rho \vec{\mathrm{v}} \cdot d\vec{S} . \tag{3.18}$$

From (3.17) and (3.18) one obtains:

$$-\int_{(V)} \frac{\partial \rho}{\partial t}\, dV = \oint_{(S)} \rho \vec{\mathrm{v}} \cdot d\vec{S} . \tag{3.19}$$

Using the Stokes theorem one gets:

$$-\int_{(V)} \frac{\partial \rho}{\partial t}\, dV = \int_{(V)} \nabla \cdot (\rho \vec{\mathrm{v}})\, dV \tag{3.20}$$

and

$$\int_{(V)} (\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{\mathrm{v}})\, dV = 0 . \tag{3.21}$$

From (3.21) it follows:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{\mathrm{v}}) = 0 . \tag{3.22}$$

This is the continuity equation. If the fluid is incompressible ($\rho = const.$) the gradient of the density of an incompressible flow is zero and the partial derivative of density with respect to time is zero and (3.22) becomes:

$$\nabla \cdot \vec{\mathrm{v}} = 0 . \tag{3.23}$$

The vector

$$\vec{j} = \rho \vec{\mathrm{v}} \tag{3.24}$$

is the density vector of the mass flux.

An incompressible flow can be described by using a velocity field which is solenoidal. A solenoidal field, has a zero divergence, and also has a non-zero curl (rotational component). If an incompressible flow also has a curl of zero, so that it is also irrotational, then the velocity field is actually Laplacian (Laplacian vector field is a vector field which is both irrotational and incompressible). If the field is denoted as $\vec{v}$, then it is described by the following differential equations:

$$\nabla \times \vec{v} = 0 \tag{3.25}$$

and

$$\nabla \cdot \vec{v} = 0. \tag{3.26}$$

*Applications**

The most common flow meter is the *magnetic flow meter*. A magnetic field is applied to the metering tube. A potential difference proportional to the flow velocity perpendicular to the flux lines appeared. The physical principle at work is electromagnetic induction. For the magnetic flow meter a conducting fluid is needed (water that contains ions) and an electrical insulating pipe surface (a rubber-lined steel tube).

Also, there are ultrasonic flowmeters.

3.1.4.2 Bernoulli's Equation

In fluid flow, an increase in velocity occurs simultaneously with decrease in pressure. This is *Bernoulli's principle*. This principle is a simplification of Bernoulli's equation which states that the sum of all forms of energy in a fluid that flows along an enclosed path, a streamline has the same value at any two points in that path.

Bernoulli's principle can be used to analyze the Venturi effect (is a special case of Bernoulli's principle) in the case of fluid or air flow through a tube or pipe that has a constriction in it. In the restriction the fluid is speeding up. Also, is reducing its pressure and producing a partial vacuum via the Bernoulli effect. This is used in carburetors and elsewhere. In a carburetor, air is passed through a Venturi tube for increasing its speed and decreasing its pressure. The low pressure air is routed over a tube leading to a fuel bowl. The low pressure sucks the fuel into the airflow, and in this way the combined fuel and air is sent to the engine. The decreasing of pressure is proportional to the rate of air flow squared.

We establish Bernoulli's equation. Upon an infinitesimal volume acts the volume force $d\vec{G} = dm\vec{g}$ and it follows:

$$\vec{f} = \frac{d\vec{G}}{dm} = \vec{g}\,. \tag{3.27}$$

We have $\vec{g} = \vec{g}(0,0,-g)$ and with Euler's equation $\frac{d\vec{\mathrm{v}}}{dt} = -\frac{1}{\rho}\nabla p + \vec{f}$ we obtain the components:

$$\frac{d\mathrm{v}_{\mathrm{x}}}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \frac{d\mathrm{v}_{\mathrm{y}}}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \frac{d\mathrm{v}_{\mathrm{z}}}{dt} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z}. \tag{3.28}$$

We multiply the first equation (3.28) by dx, the second by dy and the third by dz, we make the sum and one gets:

$$d\mathrm{v}_{\mathrm{x}}\frac{dx}{dt} + d\mathrm{v}_{\mathrm{y}}\frac{dy}{dt} + d\mathrm{v}_{\mathrm{z}}\frac{dz}{dt} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right) - g\,dz\,, \tag{3.29}$$

or

$$\mathrm{v}_{\mathrm{x}}d\mathrm{v}_{\mathrm{x}} + \mathrm{v}_{\mathrm{y}}d\mathrm{v}_{\mathrm{y}} + \mathrm{v}_{\mathrm{z}}d\mathrm{v}_{\mathrm{z}} = -\frac{1}{\rho}dp - g\,dz\,. \tag{3.30}$$

We have for the speed $\mathrm{v}^2 = \mathrm{v}_{\mathrm{x}}^2 + \mathrm{v}_{\mathrm{y}}^2 + \mathrm{v}_{\mathrm{z}}^2$ and $\mathrm{d}(\mathrm{v}^2) = 2(\mathrm{v}_{\mathrm{x}}d\mathrm{v}_{\mathrm{x}} + \mathrm{v}_{\mathrm{y}}d\mathrm{v}_{\mathrm{y}} + \mathrm{v}_{\mathrm{z}}d\mathrm{v}_{\mathrm{z}})$ and (3.30) becomes:

$$\frac{1}{2}\mathrm{d}(\mathrm{v}^2) = -\frac{1}{\rho}dp - g\,dz\,. \tag{3.31}$$

Also, one obtains:

$$\mathrm{d}(\frac{1}{2}\mathrm{v}^2)+\rho g z+p=0. \qquad (3.32)$$

From (3.32) it follows the *Bernoulli's* law for an ideal, incompressible flow in a uniform gravitational field that is given by:

$$\frac{1}{2}\rho\mathrm{v}^2+\rho g z+p=const. \qquad (3.33)$$

In (3.33) p is the *static pressure*, $\frac{1}{2}\rho\mathrm{v}^2$ is the *dynamic pressure* and $\rho g z$ is the *hydrostatic pressure*.

In the case of two arbitrary cross sections one obtains:

$$\frac{1}{2}\rho\mathrm{v}_1^2+\rho g z_1+p_1=\frac{1}{2}\rho\mathrm{v}_2^2+\rho g z_2+p_2, \qquad (3.34.a)$$

where v_1 is the speed at the S_1 surface and v_2 is the speed at the S_2 surface and $z_1=h_1$ and $z_2=h_2$.

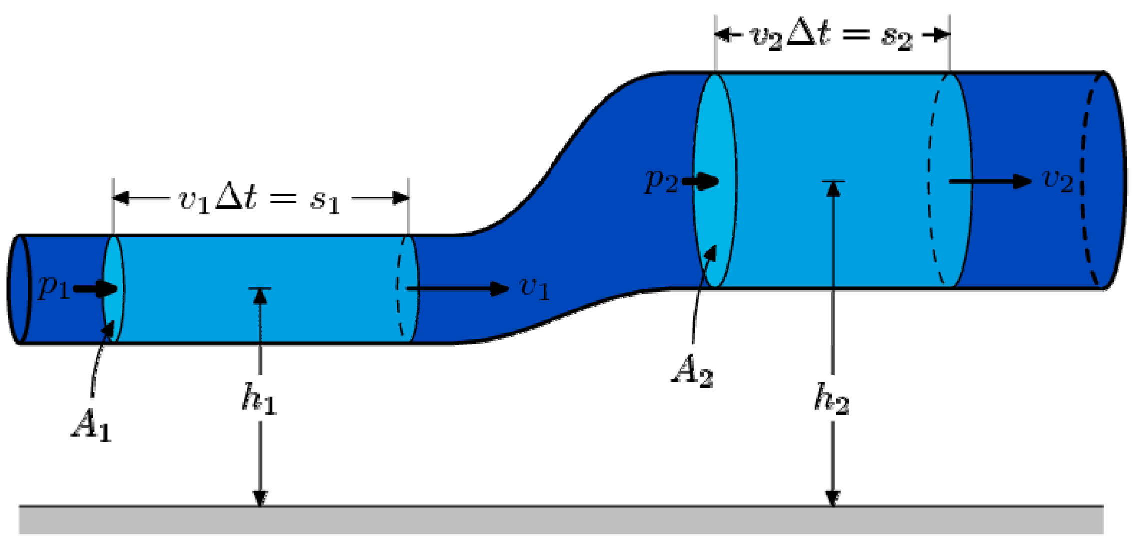


Fig. 1.39

Bernoulli's law*

For $v_2 << v_1$ one gets:

(3.34.b)

and the speed is given by:

$$v = \sqrt{2g(z_1 - z_2)}\,. \qquad (3.34.b)$$

The fluid has the same velocity as it falls down free from a height $z_1 - z_2$ (Torricelli law).

The Bernoulli equation for incompressible fluids can be derived applying the law of conservation of energy in two sections along a streamline, without taking into account the viscosity, compressibility, and thermal effects.

3.1.5 Viscosity. Laminar Flow. Navier-Stokes Equations. Transport Phenomena. Turbulence. Applications.

3.1.5.1 Viscosity. Laminar Flow.

In a fluid there are interactions between the infinitesimal volumes or between the fluid and the vessel. In these cases appear frictional forces between the layers of fluids or between the fluid and vessel.

Viscosity is a measure of the resistance of a fluid to deform under shear stress. It can be associated to thickness, or resistance to pouring. Viscosity describes a fluid's internal resistance to flow. It is a measure of fluid friction. Examples: water is thin, because it has a lower viscosity, while vegetable oil is thick because it has a higher viscosity. All real fluids (except superfluids) presents a resistance to shear stress. The ideal fluid is the fluid that has no resistance to shear stress.

Some of the real fluids are superfluids. The phase of matter characterized by the complete absence of viscosity is called superfluidity. Placed in a closed loop, a superfluid can flow endlessly without friction. In 1937 Pyotr Leonidovich Kapitsa, John F. Allen, and Don Misener discovered the superfluidity. The study of superfluidity is called quantum hydrodynamics. Lev Landau created the phenomenological theory of superfluidity in helium-4, and Nikolay Bogoliubov first suggested simple microscopical theory. When a wind is blowing over the surface of the ocean a shear stress is applied to the fluid. The fluid flows, and continues to flow while the stress is applied. When the stress is removed, in general, the flow decays due to internal dissipation of energy. Also, the thicker is the fluid, the greater is its resistance to shear stress and the more rapid the decay of its flow. In reality, when a fluid flows, the layers move at different velocities (there is not a single value for the velocity). The layers act one upon the other. The fluid's viscosity (thickness) arises from the shear stress between the layers that ultimately oppose any applied force. In the figures bellow we have two cases of laminar shear. The *laminar flow* is the flow for which the layers of fluid are parallel one to other.

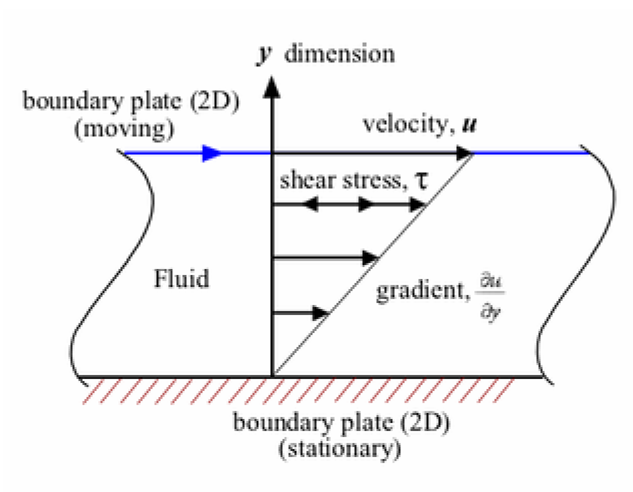


Fig. 1.40

Laminar shear of fluid between two plates. Friction between the fluid and the moving boundaries causes the fluid to shear. The force required for this action is a measure of the fluid's viscosity.*

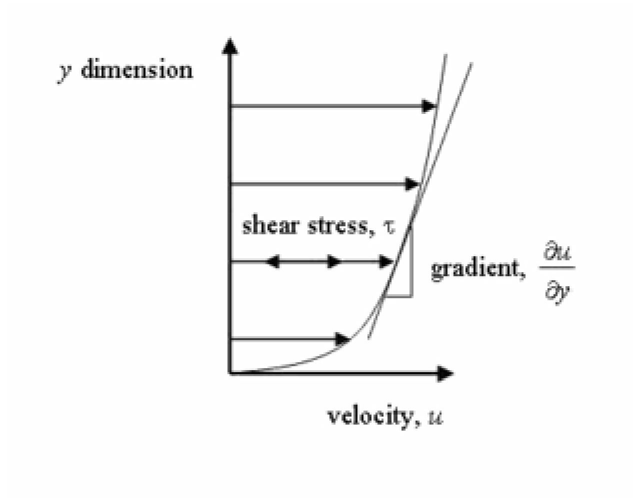


Fig. 1.41

Laminar shear, the non-linear gradient, is a result of the geometry the fluid is flowing through (a pipe)*

The viscosity is characterized by the *coefficient of viscosity* η, *viscosity*, or *dynamic viscosity*. The coefficient of viscosity η depends on the fluid and temperature. For liquids, the dynamic viscosity decreases with the increasing of temperature and for gases increases with the increasing of temperature. The dynamic viscosity is implied in the expression of the shear stress and of the frictional force. For straight, parallel and uniform flow, the shear stress, τ, between layers is proportional to the velocity gradient, $\partial \mathrm{v}/\partial y$, ($\partial \mathrm{u}/\partial y$ in the figure above), in the direction perpendicular to the layers. We have for the relative motion of the layers the shear stress $\tau = \eta \frac{d\mathrm{v}}{dy}$. Many fluids, water and most gases (Newtonian fluids), satisfy Newton's criterion (the relationship between shear stress and velocity gradient is of simple linearity). For the Non-Newtonian fluids there is a more complicated relationship between shear stress and velocity gradient than simple linearity.

For the Newtonian fluids the frictional force is given by:

$$F_r = -\eta \frac{d\mathrm{v}}{dy} S, \quad (3.35)$$

where S is the common area of two adjacent layers. In SI physical unit of dynamic viscosity η is the pascal-second (Pa·s), which is identical to 1 kg·m^{-1}·s^{-1}.

For gases viscosity is due principally to the molecular diffusion that transports momentum between layers of flow. From the kinetic theory of gases we have that viscosity is independent of pressure and increases as temperature increases.

For liquids it is known that the additional forces between molecules become important. This implies the existence of an additional contribution to the shear stress though the exact mechanics of this are still controversial. Thus, in liquids viscosity does not depend on pressure (except at very high pressure) and tends to fall as temperature increases (for example, water viscosity goes from 1.79 cP to 0.28 cP in the temperature range from 0°C to 100°C).

The values of dynamic viscosities of liquids are typically several orders of magnitude higher than the value of dynamic viscosities of gases.

Some dynamic viscosities of Newtonian fluids are given in the table below, for gases and liquids.*

gases (at 0 °C)

	viscosity (Pa·s)
hydrogen	8.4×10^{-6}
air	17.4×10^{-6}
xenon	21.2×10^{-6}

and liquids (at 25 °C)

	viscosity (Pa·s)
ethanol	[a] 1.074×10^{-3}
acetone	[a] 0.306×10^{-3}
methanol	[a] 0.544×10^{-3}
propanol	[a] 1.945×10^{-3}
benzene	[a] 0.604×10^{-3}

water	[a] 0.890×10^{-3}
nitrobenzene	[a] 1.863×10^{-3}
mercury	[a] 1.526×10^{-3}
sulfuric acid	[a] 24.2×10^{-3}
glycerol	[a] 934×10^{-3}
olive oil	81×10^{-3}
castor oil	0.985
molten polymers	10^{3}
pitch	10^{11}
glass	10^{40}

The viscometer or viscosimeter is used for measure the viscosity, typically at 25°C (standard state). For some fluids, it is a constant over a wide range of shear rates. The viscometer is also used to measure the flow parameters of a fluid. The classical method of measuring due to Stokes, consisted of measuring the time for a fluid to flow through a capillary tube. The glass tube viscometer was refined by Cannon, Ubbelohde and others and is the best method for the standard determination of the viscosity of water. The viscosity of water at 25 degrees Celsius is 0.890 mPa·s or 1.002 mPa·s at 20 degrees Celsius. The required force for rotating a disk or bob in a fluid at a known speed can be measured with the Brookfield-type viscometer. The function of other viscometer types is based on the use of bubbles, balls or other objects. Rheometers or plastometers are the viscometers that can measure fluids that have high viscosity or molten polymers.

Vibrational viscometers date back to the 1950s Bendix instrument, which is of a class which operates by measuring the damping of an oscillating electromechanical resonator immersed in a fluid whose viscosity is to be determined. The resonator generally oscillates in torsion or transversely. A higher value of the viscosity determines a larger damping imposed on the resonator. The resonator's damping may be measured using one of the methods:

1. Measuring the power input that is needed for keeping the oscillator vibrating at constant amplitude. A higher value of the viscosity determines a greater value of the power that is needed to maintain the amplitude of oscillation.

2. Evaluating the decay time of the oscillation once the excitation is switched off. A higher value of the viscosity implies a faster signal decays.

3. The frequency of the resonator can be measured as a function of phase angle between excitation and response waveforms. The frequency change for a given phase changes increases for a higher value of the viscosity

3.1.5.2 Navier-Stokes Equations.

For ideal fluids the motion is described by the Euler equation $\frac{d\vec{v}}{dt} = -\frac{1}{\rho}\nabla p + \vec{f}$. In the case of the fluids which has viscosity we have to add in the Euler equation a term that corresponds to the internal frictional forces (viscous forces) between the layers of fluid. The Navier Stokes equations govern the fluid hydrodynamics. We give the Navier-Stokes equations which are a set of equations that describe the motion of fluid substances such as liquids and gases. These equations establish that changes in momentum (acceleration) of fluid particles are the product of changes in pressure and dissipative viscous forces (similar to friction) acting inside the fluid. These viscous forces are due to the molecular interactions and control the *stickiness* (viscosity) of a fluid. The Navier-Stokes equations represent the dynamical statement of the balance of forces acting at any given region of the fluid. Some applications of the Navier Stokes equations are: modeling of the weather, ocean currents, water flow in a pipe, motion of stars inside a galaxy, flow around an airfoil (wing). Also, they play an important role in the design of aircraft and cars, the study of blood flow, the design of power stations, the analysis of the effects of pollution, etc.

The Navier-Stokes equations are differential equations which describe the motion of a fluid. The Navier-Stokes equations for the ideal fluid with zero viscosity establish that acceleration (the rate of change of velocity) is proportional to the derivative of internal pressure.

The flow is assumed to be differentiable and continuous, allowing the conservation laws to be expressed as partial differential equations. In the case of incompressible flow (constant density), the variables to be solved for are the velocity components and the pressure.

For the motion of the fluid against the Ox axis and with $dm = \rho dV = \rho S dz$ one has:

$$(f_r)_x = \frac{(dF_r)_x}{dm} = \frac{1}{\rho S}\frac{(dF_r)_x}{dz} = -\frac{1}{\rho}\eta\frac{\partial^2 \mathrm{v_x}}{\partial z^2}. \tag{3.36}$$

If we have a velocity gradient against an arbitrary direction it one gets:

$$(f_r)_x = -\frac{1}{\rho}\eta(\frac{\partial^2 \mathrm{v_x}}{\partial x^2} + \frac{\partial^2 \mathrm{v_x}}{\partial y^2} + \frac{\partial^2 \mathrm{v_x}}{\partial z^2}) \equiv -\frac{1}{\rho}\eta\nabla^2 \mathrm{v_x}. \tag{3.37}$$

For the components $(f_r)_y$ and $(f_r)_z$ of the frictional force it follows:

$$(f_r)_y = -\frac{1}{\rho}\eta(\frac{\partial^2 \mathrm{v_y}}{\partial x^2} + \frac{\partial^2 \mathrm{v_y}}{\partial y^2} + \frac{\partial^2 \mathrm{v_y}}{\partial z^2}) \equiv -\frac{1}{\rho}\eta\nabla^2 \mathrm{v_y} \tag{3.38}$$

and

$$(f_r)_z = -\frac{1}{\rho}\eta(\frac{\partial^2 \mathrm{v_z}}{\partial x^2} + \frac{\partial^2 \mathrm{v_z}}{\partial y^2} + \frac{\partial^2 \mathrm{v_z}}{\partial z^2}) \equiv -\frac{1}{\rho}\eta\nabla^2 \mathrm{v_z}. \tag{3.39}$$

We can write the last three equations using vectors and one gets:

$$\vec{f}_r = -\frac{1}{\rho}\eta\nabla^2\vec{\mathrm{v}} \equiv -\frac{1}{\rho}\eta\Delta\vec{\mathrm{v}}. \tag{3.40}$$

The equation of motion for a fluid with viscosity is given by:

$$\frac{d\vec{\mathrm{v}}}{dt} = -\frac{1}{\rho}\nabla p + \vec{f} + \frac{1}{\rho}\eta\Delta\vec{\mathrm{v}}. \tag{3.41}$$

Also, one has:

$$\frac{\partial\vec{\mathrm{v}}}{\partial t} + (\vec{\mathrm{v}}\cdot\nabla)\vec{\mathrm{v}} = -\frac{1}{\rho}(\nabla p - \eta\Delta\vec{\mathrm{v}}) + \vec{f}. \tag{3.42.a}$$

We can write (3.42.a) as it follows:

$$\rho(\frac{\partial\vec{\mathrm{v}}}{\partial t} + \vec{\mathrm{v}}\cdot\nabla\vec{\mathrm{v}}) = -\nabla p + \eta\Delta\vec{\mathrm{v}} + \vec{f}, \tag{3.42.b}$$

where:

- $\rho(\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\nabla\vec{v})$ represents the inertia;

- $-\nabla p$ is the pressure gradient;

- $\eta\Delta\vec{v}$ is the viscosity (viscous stresses in the fluid) and the viscosity is represented by the vector Laplacian of the velocity field;

- $\vec{f}$ represents the effect due to other forces.

Equation (3.41) and (3.42.a) are the Navier-Stokes equations. Together with the continuity equation determines the velocity field $\vec{v}(\vec{r},t)$ and the pressure field $p(\vec{r},t)$ of a fluid for suitable boundary conditions. In the case of compressible flow the density becomes another unknown of the system, and can be determined supplementing the system with an equation of state. An equation of state usually involves the temperature of the fluid, so that the equation for conservation of energy must also be solved, coupled with the previous ones. These equations are non-linear, and analytical solutions in closed form are known only for cases with very simple boundary conditions.

3.1.6 Turbulence

Turbulence is the time dependent chaotic behavior that appears in many cases in fluid flows.

In fluid dynamics, *potential flow* or *irrotational flow* (of incompressible fluids) is steady flow defined by the equations:

$$\nabla\times\vec{v}=0 \tag{3.43}$$

and

$$\nabla \cdot \vec{v} = 0. \tag{3.44}$$

First equation means zero rotation and second equation implies zero divergence that equals to volume conservation. Also, one gets:

$$\vec{v} = \nabla\Phi, \tag{3.45}$$

where Φ is the potential. The potential flow has many applications in aircraft design.

The equations above imply $\nabla^2\Phi = 0$, or Laplace's equation, holds. Solutions of Laplace's equation are called harmonic functions. These equations, the Navier-Stokes equations and the Euler equations, can be used to calculate solutions to many practical flow situations.

It is obviously that the potential flow does not contain all the characteristics of flows that are encountered in real situations. If the flow is potential this implies the exclusion of turbulence, which is commonly encountered in nature. The opinion of Richard Feynman is that potential flow is so unphysical that the only fluid to obey the assumptions was dry water.

Also, potential flow it is not properly to describe the behaviour of flows that include a boundary layer. The boundary layer is that layer of fluid in the immediate vicinity of a bounding surface. In the atmosphere the boundary layer is the air layer near the ground that is affected by diurnal heat, moisture or momentum (in transfer ring with the Earth surface). The part of the flow that is close to the wing represents the boundary layer for an aircraft wing. In the field region in which all changes occur in the flow pattern it appears the Boundary layer effect. The boundary layer is able to modify surrounding non viscous flow. Also, the boundary layer is due to viscous forces.

The simple potential flows (elemental flows) such as the free vortex and the point source have analytical solutions. These solutions can be superposed and in this way can be created more complex flows. These flows satisfy a number of boundary conditions.

Any streamline can be replaced by a solid boundary with no change in the flow field, and this is because of the absence of boundary layer effects. This technique is used in many aerodynamic design approaches.

In fluid mechanics there are two limiting vortex cases. These are the free (irrotational) vortex, and the forced (rotational) vortex.

In the free (irrotational) vortex the tangential velocity v varies inversely as the distance r from the centre of rotation. In this case the angular momentum is constant. This implies that the vorticity is zero everywhere except for a singularity at the centre-line.

The tangential velocity is given by:

$$\mathrm{v} = \frac{\Gamma}{2\pi \mathrm{r}}, \tag{3.46}$$

where Γ is the vortex strength.

In the forced vortex the fluid rotates as a solid body with no shear. We can realized the forced vortex by placing a dish of fluid on a rotating turntable.

The tangential velocity is given by:

$$\mathrm{v} = \omega r, \tag{3.47}$$

where ω is the angular velocity and r is the radial distance from the center of the vortex.

Vortices appear in turbulent flow. Any circular or rotary flow that possesses *vorticity* represents a vortex. The vorticity is a vector and is given by the circulation per unit area at a point in the flow field. The movement of a fluid is *vortical* if the fluid moves around in a circle, or in a helix, or if it tends to spin around some axis. Such motion can also be called solenoidal. Vorticity is given by:

$$\vec{\omega} = \nabla \times \vec{\mathrm{v}}, \tag{3.48}$$

where $\vec{\mathrm{v}}$ is the fluid velocity. It can be also expressed by the circulation per unit area at a point in a fluid flow field. It is a vector that has the direction along the axis of the fluid's rotation.

Turbulence or *turbulent flow* is a flow regime characterized by chaotic property changes. There are rapid variation of pressure and velocity in space and time.

We can classify the fluid flows in laminar and turbulent flows with the *Reynolds number*. The *Reynolds number* is given by:

$$\mathrm{Re} = \frac{\rho \mathrm{v} L}{\mu}, \tag{3.49}$$

and is equal to the ratio of inertial forces $\rho \mathrm{v}$ to viscous forces $\frac{\mu}{L}$. It is dimensionless. The transition between laminar and turbulent flow is indicated by a critical Reynolds number Re_{crt}. For circular pipe flow, a Reynolds number above about 2300 will be turbulent. For laminar flow we have low Reynolds numbers, and the viscous forces are dominant. It is characterized by smooth, constant fluid motion. For turbulent flow we have high Reynolds numbers and is dominated by inertial forces, producing random eddies, vortices and other flow fluctuations.

Some characteristics of the turbulent flow are:

- there are not streamlines and the velocity field $\vec{\mathrm{v}}$ is not a continue function of any point;

- the turbulent flow is not stationary and we have $\frac{\partial \vec{\mathrm{v}}}{\partial \mathrm{t}} \neq 0$, which means that there are rapid variation of velocity in space and time;

- the frictional (viscous) force is not proportional to the velocity v, but is proportional to v^n, where $n > 1$.

Examples of vortex:

- vortex created by the passage of an aircraft wing;
- the spiraling motion of air or liquid around a center of rotation;
- circular current of water of conflicting tides form vortex shapes;
- the atmospheric phenomenon of a whirlwind (takes the form of a helix, column, or spiral) or a tornado (develop from severe thunderstorms, usually spawned from squall lines (organized lines of thunderstorms) and super cell thunderstorms (a severe thunderstorm), though they sometimes happen as a result of a hurricane);
- vortex usually formed as water goes down a drain, as in a sink or a toilet. This occurs in water as the revolving mass forms a whirlpool (a large, swirling body of water produced by ocean tides). This whirlpool is caused by water flowing out of a small opening in the bottom of a basin or reservoir. This swirling flow structure within a region of fluid flow opens downward from the water surface.

Fig. 1.42

Vortex created by the passage of an aircraft wing, revealed by coloured smoke*

Multiple vortices appear in the case of stronger tornadoes, the tornado contains several vortices that rotates around and inside of and part of the main vortex. Also, these stronger tornadoes have many columns of violently spinning air that rotate around a common center. A satellite tornado is a weak tornado which ia accompanying a large, strong tornado, many times it ends in no more than a minute.

Fig. 1.43

Union City, Oklahoma tornado (1973)*

A tropical cyclone represents a storm system that is characterized by a closed circulation around a centre of low pressure. The tropical cyclone is generated by the heat transferred by the air that rises and condenses. Tropical cyclones are called tropical depression, tropical storm, hurricane and typhoon, this classification being in strong connection with their strength and location.

Fig. 1.44

Cyclone Catarina, a rare South Atlantic tropical cyclone viewed from the International Space Station on March 26, 2004*

A whirlpool consists of a large, swirling body of water that is generated by ocean tides.

Fig. 1.45

Saltstraumen a strong tidal current located some 30 km east of the city of Bodø, Norway*

Other examples of vortex are:

- the acceleration of the electric fluid in a particular direction creates a positive vortex of magnetic fluid. As a result, a corresponding negative vortex of electric fluid is generated around it;
- a ring of smoke in the air
- Polar vortex that represents a persistent, large-scale cyclone encountered near the Earth's poles, in the middle and upper troposphere and the stratosphere;
- Sunspot that is a dark region on the Sun's surface (photosphere) that is characterized by a lower temperature than its surroundings, and has an intense magnetic activity;
- the accretion disk of a black hole or other massive gravitational source;
- Spiral galaxy that is a type of galaxy in the Hubble sequence which is characterized by a thin, rotating disk. Our galaxy, the Milky Way is of this type.

Seminar 1

1.1 Cartesian Coordinates. Dot Product, Cross Product and Vector (Differential) Operators.

The reference system can be inertial or non-inertial. The inertial reference system is the system that has a rectilinear uniform motion or is at relative rest. The non-inertial reference system has an accelerated motion. Under classical mechanics and special relativity, the inertial reference systems are considered.

To the reference system there is rigidly attached a reference frame. The motion of a moving body is univocally determined if, each moment, its coordinates are known in relation with the reference system chosen. In mechanics there are especially used:

3) Cartesian coordinate system
4) spherical coordinate system

In the Cartesian coordinate system the position of a point *P* is given by the Cartesian coordinates *x, y* and *z* (Fig. 1.46). The vector $\vec{r}$, that connects the origin with point *P*, is called position vector or radius vector.

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}. \tag{1.1}$$

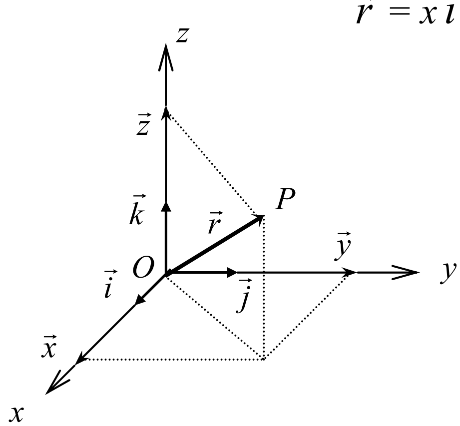


Fig. 1.46

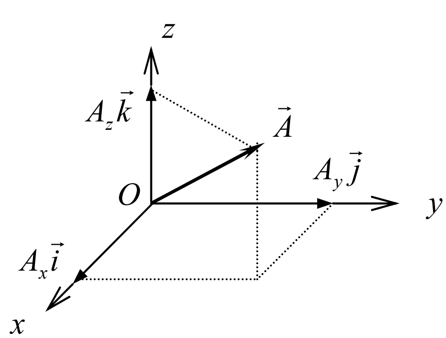


Fig. 1.47

In general, any arbitrary vector A can be written:

$$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k} \ , \tag{1.2}$$

where A_x, A_y and A_z are called the components of vector $\vec{A}$ (Fig. 1.47). According to the components, the length (*magnitude*) of vector $\vec{A}$ is:

$$A = \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}\,, \tag{1.3}$$

and $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors of the coordinate axes. One knows they satisfy the relations:

$$\left|\vec{i}\right| = \left|\vec{j}\right| = \left|\vec{k}\right| = 1; \quad \vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1;$$

$$\vec{i}\cdot\vec{j} = \vec{i}\cdot\vec{k} = \vec{j}\cdot\vec{k} = 0$$

In the spherical coordinates system the position of a point *P* is given by the spherical coordinates *r, θ, φ*.

Let us consider two vectors $\vec{A}$ and $\vec{B}$. We define the *dot (scalar) product*:

$$\vec{A}\cdot\vec{B} = AB\cos\alpha, \tag{1.4}$$

where α is the angle between the two vectors $\vec{A}$ and $\vec{B}$. In Cartesian coordinates with $\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$ and $\vec{B} = B_x\vec{i} + B_y\vec{j} + B_z\vec{k}$ we get:

$$\begin{aligned}\vec{A}\cdot\vec{B} &= (A_x\vec{i} + A_y\vec{j} + A_z\vec{k})\cdot(B_x\vec{i} + B_y\vec{j} + B_z\vec{k}) = \\ &= A_xB_x + A_yB_y + A_zB_z\end{aligned} \tag{1.5.a}$$

Also, we define for the vectors $\vec{A}$ and $\vec{B}$ the *cross product*:

$$\vec{C} = \vec{A}\times\vec{B} = \vec{n}\,AB\sin\alpha \ , \tag{1.5.b}$$

where $\vec{n}$ is an unit vector normal (perpendicular) to both $\vec{A}$ and $\vec{B}$. The cross product is defined as the vector which is normal (perpendicular) to both $\vec{A}$ and $\vec{B}$ with a magnitude equal to the area of the parallelogram they span. It has the length (*magnitude*)

$$C = AB\sin\alpha \tag{1.5.c}$$

An easy way to compute the direction of the resultant vector is the "right-hand rule." One simply points the forefinger in the direction of the first operand and the middle finger in the direction of the second operand. Then, the resultant vector is coming out of the thumb.

In Cartesian coordinates one gets:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \tag{1.6}$$

Exercise 1:

Let us consider the pairs of vectors:

1) $\vec{A} = 5\vec{i} - 4\vec{j} + 2\vec{k}$ and $\vec{B} = -2\vec{i} + 3\vec{j} + \vec{k}$,

2) $\vec{C} = 4\vec{i} - 4\vec{j} - 2\vec{k}$ and $\vec{D} = 7\vec{i} + \vec{j} - 3\vec{k}$

and compute the dot product and the cross product in these two cases.

A **vector operator** is a type of differential operator used in vector calculus.

Now, we define the vector (differential) operators: del ∇ (nabla), gradient, divergence, curl and Laplacian.

1) **Del ∇ (nabla)**

In vector calculus, **del** is a vector differential operator represented by the nabla symbol. In Cartesian coordinates is defined as:

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\,. \tag{1.7}$$

With del we define the gradient, divergence, curl and Laplacian.

2) **Gradient**

Let φ be a scalar function and we get the gradient of φ in Cartesian coordinates:

$$grad\,\varphi = \nabla\varphi = \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j} + \frac{\partial\varphi}{\partial z}\vec{k}\,. \tag{1.8}$$

The vector field $\vec{A} = grad\,\varphi$ is called *potential* (*curl-free*) *field*. Example, the static field $\vec{E} = -gradV$, where V is the electric potential $V(x, y, z)$. Also, the gravitational field is an example of such a field: gravity gradients, see in the figure below.

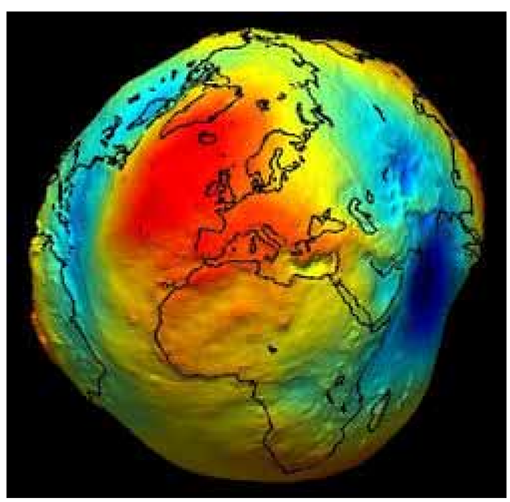


Fig. 1.48*

3) **Divergence**

Let us consider the vector $\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$ (Cartesian coordinates) and we define the divergence of $\vec{A}$:

$$\mathrm{div}\vec{A} = \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \tag{1.9}$$

The vector field that satisfies $\nabla \cdot \vec{A} = 0$ is a *solenoidal field*. Example: induction $\vec{B}$ (magnetic field). Also, the velocity field of an incompressible fluid flow is solenoidal.

Divergence Theorem (**Green-Gauss-Ostrogradski Theorem**):

$$\iiint_V \nabla \cdot \vec{A}dV = \iint_S \vec{A} \cdot d\vec{S} \tag{1.10}$$

The divergence theorem states that the flux of a vector field on a surface is equal to the triple integral of the divergence on the region inside the surface.

Exercise 2:

Suppose we wish to evaluate the flux of the vector $\vec{A}$, $\iint_S \vec{A} \cdot \vec{n}dS$ where S is the unit sphere defined by $x^2 + y^2 + z^2 = 1$ and $\vec{A} = 2x\vec{i} + y^2\vec{j} + z^2\vec{k}$ is the vector field.

4) **Curl**

In vector calculus, **curl** is a vector operator that shows a vector field's rate of rotation: the direction of the axis of rotation and the magnitude of the rotation.

Let us consider the vector $\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$ (Cartesian coordinates) and we define the curl of $\vec{A}$ (del cross $\vec{A}$) as:

$$curl\,\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}. \tag{1.11}$$

The vector field that satisfies $\nabla \times \vec{A} = 0$ is an *irrotational field*. Example: static field $\vec{E}$. In fluid mechanics, an *irrotational* field is practically synonymous with a *lamellar field*. The adjective "*irrotational*" implies that irrotational fluid flow (whose velocity field is irrotational) has no rotational component: the fluid does not move in circular or *helical* motions; it does not form *vortices*. Opposite: vortex or helices, see figures below.

Fig. 1.49*

Fig. 1.50*

Stokes-Ampère Theorem:

$$\oint_{(C)} A \cdot d\vec{l} = \iint_{S} \mathrm{rot}\,\vec{A} \cdot d\vec{S} \tag{1.12}$$

which relates the surface integral of the curl of a vector field over a surface S in Euclidean 3 space to the line integral of the vector field over its boundary.

5) **Laplacian**

The Laplace operator is a second order differential operator, defined as the divergence of the gradient:

$$\Delta = \nabla^2 = \nabla \cdot \nabla \,. \qquad (1.13)$$

The Laplacian is the sum of all the *unmixed* second partial derivatives. In the three-dimensional space the Laplacian is commonly written as

$$\Delta = \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \,. \qquad (1.13)$$

Exercise 3:

Let us consider the scalar functions:

1) $\varphi = 2xy + 4y^2 - 3\sin(z)$, 2) $\varphi' = 4x^2yz - 5yz + 7xyz$

and compute the gradient in the both cases.

Exercise 4:

Let us consider the pairs of vectors:

1) $\vec{A} = 5x\vec{i} - 4y^2\vec{j} + 2z\vec{k}$ and $\vec{B} = 2y\vec{i} + 3xyz\vec{j} + yz\vec{k}$,

2) $\vec{C} = -xz\vec{i} - 4x^2yz\vec{j} + 2xy\vec{k}$ and $\vec{D} = x\vec{i} + 2xz\vec{j} - 3xyz\vec{k}$

and compute the divergence and the curl.

Seminar 2

1. Draw the space, speed and acceleration graphs after 2 seconds from the beginning of the motion of a material point, whose acceleration is given by the ratio $a = kt$, where k = 0.1 m/s^3, considering that at the initial moment, the speed is zero, and the space is 5 m.
2. Determine the speed and space of a material point whose acceleration depends on the speed according to the law $a = -k\text{v}^2$, knowing that when $t = 0$, $\text{v} = \text{v}_0$ and $s = s_0$.
3. Consider a motion defined by the parametric equations $x = 3e^t + \frac{1}{4}$, $y = 4e^t - 1$. Determine trajectory of the material, its speed and acceleration.
4. A constant force $\vec{F}$ is acting upon a body with mass m. The motion is along axis x and the coefficient of friction is μ. At the initial moment, the velocity is $\vec{\text{v}}_0$. Determine the equation of motion if $\vec{\text{v}}_0$ is parallel to $\vec{F}$.

5) On a body with $m = 1kg$, acts the force with an instantaneous value of $\vec{F} = 2y^2\vec{i} + 3x^2\vec{j} - (x^2 + y^2)\vec{k}\,(N)$. At the initial moment, the body is in the position $M(2,3,0)$ and its velocity is $\text{v}_0 = 3\vec{j} + \vec{k}\,(m/s)$. When $t = 0$, determine: a) the $\vec{F}_0$ force acting on the body; b) the acceleration of the body $\vec{a}_0$; c) kinetic E_{c0} energy; d) rate of change of the kinetic energy.
6) Determine the equation of motion of a material point that is moving in the field given by $U(x) = -Ax^4$, if its energy is zero.
7) Two springs with elastic constants k_1 and k_2 are series connected and have a

fixed extremity. At the free extremity there is a body hanging. Determine the springs' E_{p1} / E_{p2} energies ratio.

Seminar 3

1) A material point describes two normal motions, given by equations $x = 10\cos 3t$ cm and $y = 10\sin 3t$ cm. Lets determine trajectory, speed and acceleration.
2) The parametric equations of a motion of a material point are given by $x = a\cos kt,\ y = \frac{g}{k^2}(1 - \cos kt)$. Lets determine trajectory and displacement if at $t = 0,\ s = 0$.
3) Knowing the speeds $\mathrm{v}_1 = 3$ cm/s and $\mathrm{v}_2 = 5$ cm/s which correspond to the displacements $x_1 = 6$ cm and $x_2 = 4$ cm, let determine: a) maximum of displacement (amplitude) and period of the harmonic oscillations, b) maximum of acceleration.
4) A body of mass $m = 150$ g is connected to a spring with constant spring $k = 50$ N/m and describes damped oscillations. After $n = 15$ oscillations the amplitude decreases and becomes $A_0 / 2{,}71$. Lets determine: a) damping ratio β, b) period T, c) damping factor δ.
5) A bob pendulum describes damped oscillations with the damping ratio $\beta = 0{,}04$. After 50 s, the energy of the bob pendulum decreased and becomes $E_1 / 7{,}39$. Let determine the length of the wire (pendulum).
6) For the damped oscillations with $\beta = 0{,}15$ lets determine the ratio $\mathrm{v}(t+T)/\mathrm{v}(t)$ and the ratio $a(t+T)/a(t)$.

Seminar 4

1) A body of mass $m = 150$ g is connected to a spring with constant spring $k = 50$ N/m and describes damped oscillations. After $n = 15$ oscillations the amplitude decreases and becomes $A_0 / 2{,}71$. Lets determine: a) damping ratio β, b) period T, c) damping factor δ.
2) A bob pendulum describes damped oscillations with the damping ratio $\beta = 0{,}04$. After 50 s, the energy of the bob pendulum decreased and becomes $E_1 / 7{,}39$. Lets determine the length of the wire (pendulum).

3) We have two harmonic oscillations on the same direction described by the equations $x_1 = 0.01\sin(\frac{2\pi}{3}t + \frac{\pi}{6})$ m and $x_2 = 0.05\sin(\frac{2\pi}{3}t + \frac{\pi}{3})$ m. Let determine the equation of the combined motion.

4) Lets determine the resultant motion of a material point that is undergoing two orthogonal motions given by:

a) $x = 3\sin(\frac{\pi}{6}t + \frac{\pi}{2})$; $y = 2\sin\frac{\pi}{6}t$

b) $x = 6\sin(2\pi t + \frac{3\pi}{2})$; $y = 6\sin 2\pi t$

c) $x = 3\sin(\pi t + \frac{\pi}{6})$; $y = 4\sin(\pi t + \frac{\pi}{6})$.

5) Lets determine the trajectory of a mobil that is undergoing two orthogonal harmonic oscillations given by $x = \sin(\frac{\pi}{2}t + \frac{\pi}{2})$ and $y = 2\cos\frac{\pi}{4}t$.

Seminar 5

1) A body is hanging on a spring and has the period $2s$. Upon it are acting a sinusoidal force with amplitude $F = 0.1N$ and a frictional force proportional to velocity. Knowing that at the resonance of speeds the amplitude of oscillations is $A_0 = 5\,cm$, lets determine the damper constant γ.

2) A body of mass $m = 250\,g$ is undergoing damped oscillations with the damping factor $\delta = 0.785\,s^{-1}$ and the period of the simple harmonic oscillations is $T_0 = \frac{2}{\sqrt{3}}s$. The body is undergoing driven oscillations when an external force $F = 0.1\sin 2\pi t\,(N)$ is acting upon it. Lets write the law of motion for the driven oscillations.

3) A source placed in an elastic medium emits plane waves $y = 0.25\sin 100\pi t\,(mm)$. The wavelength of the longitudinal waves is $\lambda = 10\,m$. Lets determine:

a) the time that a point placed at the distance $x_1 = 8\,m$ far from the source needed to begins to oscillate and the phase difference between the oscillation of this point and source

b) the distance between two points which have the phase difference $\frac{\pi}{6}$

c) the phase difference between two points which are placed at $\frac{\lambda}{2}$ one from the other

4) The amplitude of a simple harmonic oscillation is $A = 10\,cm$, the frequency is $\nu = 4\,Hz$ and the velocity is $v = 100\,m/s$. Lets determine the displacement, the

velocity and acceleration of points placed at the distance $x = 75\,m$ far from the source at the moment $t = 1\,s$ after the beginning of the motion.

Seminar 6

1. A string AB with length $l = 9\ m$ is fixed to the B extremity. The B extremity oscillates transversely with amplitude $A = 5\ cm$ and frequency $\nu = 10\ Hz$. The wave velocity is $\mathrm{v} = 4{,}3\ m/s$. Lets determine the law of oscillations of a point M if the displacement MB is $x = 93{,}75\ cm$ and the positions of the troughs (minimum).

2. In a steel rod propagate longitudinal waves. The diameter of the rod is $d = 4\ mm$, amplitude of oscillations $A = 0{,}1\ mm$ and frequency $\nu = 10\ Hz$. Let determine:

a) equation of the wave

b) wave energy density

c) average flux of energy

We know the Young's modulus $E = 2 \cdot 10^{11}\ N/m^2$ and linear density $\rho = 7{,}8 \cdot 10^3\ kg/m^3$.

3. Discussions about the equation of the wave, wave energy density and average flux of energy.

Seminar 7

1. In a vessel which contains mercury is putted a vertical tube with cross section $S = 10\ cm^2$. The tube has a piston with mass $m = 2\ kg$. Initially, the piston is at the same level with the mercury contained in the vessel. The piston is displaced above at $h = 0{,}2\ m$. Lets determine:

a) the force that acts upon the piston when the height of the mercury in the tube is $h = 0{,}2\ m$. The linear density is $\rho = 13580\ kg/m^3$.

b) the mechanical work done when the piston have been displaced at $h = 0{,}2\ m$

2. A syringe has the cross section of the piston $S_1 = 1{,}2\ cm^2$ and the cross section of the needle $S_2 = 0{,}8\ mm^2$. The piston is displaced with $l = 6\ cm$. In the syringe here is a liquid with linear density $\rho = 1{,}4 \cdot 10^3\ kg/m^3$. Upon the piston acts the constant force $F = 47{,}258\ mN$. Lets determine the required time for putting out the liquid from the syringe.

3. Discussion about the fundamental law of hydrodynamics, hydrostatic pressure and Bernoulli's equation.

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